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**A complete proof of the Poincaré and geometrization conjectures – application of the Hamilton-Perelman theory of the Ricci flow.** (English) Zbl 1200.53057

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The authors present a complete proof of the conjectures stated in the title, based primarily on numerous deep works of *R. S. Hamilton* on the Ricci flow, beginning with his fundamental paper [ J. Differ. Geom. 17, No. 2, 255–306 (1982; Zbl 0504.53034)], and the remarkable recent work of *G. Perelman* [arXiv e-print service, Cornell University Library, Paper No. 0211159, electronic only (2002; Zbl 1130.53001); arXiv e-print service, Cornell University Library, Paper No. 0303109, electronic only (2003; Zbl 1130.53002); arXiv e-print service, Cornell University Library, Paper No. 0307245, electronic only (2003; Zbl 1130.53003)]. The Poincaré and geometrization conjectures have a long history. Numerous fundamental contributions have been made by many mathematicians, and the geometrization conjecture has been verified for several important classes of 3-manifolds.

Geometrization. Thurston's geometrization conjecture states roughly that every compact, orientable 3-manifold can be canonically decomposed into finitely many pieces, each of which admits a unique geometric structure as a complete, locally homogeneous, Riemannian manifold. This decomposition is achieved by cutting the manifold along certain 2-spheres and tori.

A classical theorem of Kneser states that every compact, orientable 3-manifold $M$ has a connected sum decomposition as a finite sum of orientable prime 3-manifolds

$$M = (\#_j X_j) \# (\#_k Y_k) \# \left(\#_l(\mathbb{S}^2 \times \mathbb{S}^1)\right). \tag{1}$$

A compact, orientable 3-manifold $N$ is said to be prime if it is not diffeomorphic to $\mathbb{S}^3$ and if a connected sum decomposition $N = N_1 \# N_2$ is possible only if one of the factors $N_1, N_2$ is diffeomorphic to $\mathbb{S}^3$. In the decomposition (1) each $X_j$ is irreducible with finite fundamental group and universal cover a homotopy 3-sphere, and each $Y_k$ is irreducible with infinite fundamental group and contractible universal cover. Irreducible means that every separating embedded 2-sphere bounds a 3-ball. $\mathbb{S}^2 \times \mathbb{S}^1$ is prime but not irreducible, and is the only such orientable 3-manifold.

Thurston's geometrization conjecture can be stated as follows. Let $M$ be a closed, orientable, prime 3-manifold. Then there is an embedding of a disjoint finite union of incompressible 2-tori $\coprod_i T_i$ such that every component of the complement is geometrizable in the sense that it admits a locally homogeneous, complete, Riemannian metric of finite volume. Incompressible means that the inclusion map induces an injection of fundamental groups. The existence of the torus decomposition was proved by *W. H. Jaco* and *P. B. Shalen* [Mem. Am. Math. Soc. 220, 192 p. (1979; Zbl 0415.57005)] and *K. Johannson* [Homotopy equivalences of 3-manifolds with boundaries. Lecture Notes in Mathematics. 761. Berlin-Heidelberg-New York: Springer-Verlag. (1979; Zbl 0412.57007)].

The universal cover $\widetilde{N}$ of each component $N$ in the geometrization conjecture carries a complete, homogeneous metric in which the covering transformations are isometries, and $\widetilde{N}$ is said to be the geometry on which $N$ is modelled. Only eight possible homogeneous geometries can arise in the geometrization conjecture. The three constant curvature ones are the sphere $\mathbb{S}^3$, Euclidean space $\mathbb{R}^3$, and hyperbolic space $\mathbb{H}^3$, having curvature $+1$, $0$ and $-1$ respectively. Then there are the product metrics on $\mathbb{S}^2 \times \mathbb{R}^1$ and $\mathbb{H}^2 \times \mathbb{R}^1$. The remaining three are $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the universal cover of $\mathrm{SL}(2,\mathbb{R})$, which can also be described as the universal cover of the unit tangent bundle to $\mathbb{H}^2$ with the induced metric Nil, the Heisenberg group of upper triangular $3 \times 3$ matrices with diagonal entries 1, which can also be thought of as $\mathbb{R}^3$ with the multiplication $(x,y,z) \cdot (x',y',z') = (x+x', y+y', z+z'+xy')$ and the left invariant metric $ds^2 = dx^2 + dy^2 + (dz - xdy)^2$ and Sol, which can be thought of as $\mathbb{R}^3$ with the multiplication

$$(x,y,z) \cdot (x',y',z') = (x + e^{-z}x', y + e^{z}y', z + z')$$

and the left invariant metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$.

The geometrization conjecture contains as a very special case the Poincaré conjecture: a closed 3-manifold

$\Sigma$ with trivial fundamental group is homeomorphic to the 3-sphere $\mathbb{S}^3$. The above connected sum and torus decompositions are trivial in this case, so by the geometrization conjecture $\Sigma$ has a locally homogeneous metric. Since $\pi_1(\Sigma)$ is trivial, the homogeneous model for $\Sigma$ is $\Sigma$ itself, which is compact. The only compact homogeneous model is $\mathbb{S}^3$, and therefore $\Sigma$ is diffeomorphic to $\mathbb{S}^3$.

Ricci flow. The Ricci flow is the system of equations given in local coordinates by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{2}$$

where $g_{ij}$ is the Riemannian metric and $R_{ij}$ is its Ricci curvature. It was introduced by Hamilton in 1982 to study compact 3-manifolds with positive Ricci curvature. The Ricci flow is a geometrically natural flow for metrics. Under the Ricci flow the Riemann curvature tensor $R_{ijkl}$ evolves by a system of nonlinear reaction-diffusion equations

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + Q_{ijkl}, \tag{3}$$

where $\Delta$ is the Laplacian with respect to the evolving metric and $Q_{ijkl}$ is quadratic in the curvature tensor. The diffusion term tends to distribute the curvature uniformly over the manifold, while the reaction term can force singularities to develop. From (2) it is evident that the Ricci flow shrinks metrics in the direction of positive Ricci curvature and expands them in the direction of negative Ricci curvature. The fixed points of the Ricci flow are (up to rescaling) metrics of constant Ricci curvature, which in three dimensions implies constant sectional curvature.

In his 1982 paper Hamilton showed that for any initial metric $g_0$ of positive Ricci curvature on a compact 3-manifold $M$, the Ricci flow has a smooth solution $g(t)$ on a time interval $[0,T)$ with $T < \infty$, and as $t \to T$, the diameter of $(M, g(t))$ shrinks to zero. After rescaling to keep the volume constant, the normalized metrics $\tilde{g}(t)$ converge to a metric of positive constant sectional curvature. The underlying manifold is thus diffeomorphic to $\mathbb{S}^3$ or to a quotient $\mathbb{S}^3/\Gamma$ for a finite subgroup $\Gamma$ of SO(4) acting freely on $\mathbb{S}^3$, by a classical result of H. Hopf. In particular, this implies the Poincaré conjecture for homotopy 3-spheres admitting a metric of positive Ricci curvature.

It was soon realized that the Ricci flow could provide an approach to the geometrization conjecture. Given any compact 3-manifold, one endows it with an arbitrary smooth (suitably normalized) initial Riemannian metric and evolves it by the Ricci flow (2). This has a unique solution for a short time; furthermore, the solution $g(t)$ exists on a maximal time interval $[0,T)$ where either $T = \infty$, or $T < \infty$ and the curvature becomes unbounded as $t \to T$. In the latter case the solution $g(t)$ is said to develop a singularity at time $T$.

If the flow develops a singularity, one tries to extract enough information about its structure so that geometric surgery can be used to cut off the singularity in a controlled way; that is, one knows how the topology changes as a result of removing the singularity. After the surgery one can then continue the flow until the next singularity develops; this is then removed in a similar way and the flow is continued. If one can show that only finitely many surgeries occur in any finite time interval and if the topology of longtime solutions of the Ricci flow is sufficiently well understood, then one can recover the topological structure of the original manifold. This approach became known as Hamilton's program.

Hamilton proved several fundamental results about the Ricci flow on 3-manifolds that give crucial information about the structure of singularities. First, from (2) one can derive evolution equations for the Riemann, Ricci and scalar curvatures, such as (3). These equations show that a lower bound on the scalar curvature $R$ is preserved by the flow (in all dimensions), and that positive Ricci curvature is preserved (in three dimensions).

Next, the Hamilton-Ivey pinching estimate states that if $g(t)$ is a solution of the Ricci flow on a compact 3-manifold, then there exist constants $c > 0$ and $C < \infty$, depending on the initial metric $g(0)$, such that if $R(x,t) \geq C$, then the minimum sectional curvature $K_{\min}$ satisfies

$$K_{\min} \geq -c \frac{R}{\log R} \text{ at } (x,t). \tag{4}$$

This has two important consequences. First, it implies that the scalar curvature controls the full Riemann curvature tensor. Second, if the metric is rescaled near a singularity so that the scalar curvature is bounded, then the limiting singularity model must have nonnegative curvature.

Finally, if $g(t)$ is a complete solution to the Ricci flow on a 3-manifold $N$ with bounded and nonnegative curvature operator (which is equivalent to nonnegative sectional curvatures in three dimensions), then the scalar curvature satisfies the following Harnack estimate: for any $x_1, x_2 \in M$ and $0 < t_1 < t_2$,

$$R(x_2, t_2) \geq \frac{t_1}{t_2} \exp\left(-\frac{d_{t_1}^2(x_1, x_2)}{2(t_2 - t_1)}\right) R(x_1, t_1), \tag{5}$$

where $d_{t_1}$ is the distance on $(M, g(t_1))$. In particular, this is true for nonnegatively curved singularity models.

Singularities. In the early 1990s Hamilton developed techniques to understand the structure of singularities for the Ricci flow on a compact 3-manifold. The singularities can be divided into three main classes according to whether they develop in finite or infinite time, and according to the rate at which the curvature blows up as $t \to T$. For the topological applications the finite time singularities are the main ones to understand.

By rescaling the solution about a sequence of points of almost maximal curvature and using a compactness result for smooth solutions to the Ricci flow with uniformly bounded curvatures and uniformly bounded injectivity radii, Hamilton showed that each finite time singularity is asymptotic to a quotient by isometries of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\Sigma \times \mathbb{R}$, where $\Sigma$ is the so-called cigar soliton $ds^2 = \frac{dx^2+dy^2}{1+x^2+y^2}$ on $\mathbb{R}^2$. Moreover, the Hamilton-Ivey estimate (4) implies that any three-dimensional singularity model $N$ must have nonnegative curvature; hence the Harnack estimate (5) holds on singularity models. Nonnegative curvature also imposes restrictions on the topology of possible complete singularity models in three dimensions. If $N$ is noncompact, then $N$ is diffeomorphic to $\mathbb{R}^3$ or $\mathbb{S}^2 \times \mathbb{R}$ or a quotient of these, while if $N$ is compact, then an extension of Hamilton's theorem for positive Ricci curvature implies that $N$ is diffeomorphic to $\mathbb{S}^3$, or to a quotient of one of the spaces $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a group of isometries acting freely in the standard metrics.

The two major obstacles that remained were the verification of the assumed injectivity radius condition, and excluding the possibility of forming a singularity modelled on the product of the cigar soliton with the real line; this type of singularity cannot be removed by surgery.

Perelman's work. Perelman showed that the Ricci flow is, in a certain sense, the gradient flow of the functional

$$\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2) e^{-f}\, dV$$

defined on the space of Riemannian metrics and smooth functions on $M$. A variational structure for the Ricci flow was not known previously. A corollary of this is that if $g_{ij}$ evolves according to the Ricci flow (2) and $f$ satisfies the backward heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R,$$

then

$$\frac{d}{dt}\mathcal{F}(g_{ij}(t), f(t)) = 2\int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f}\, dV$$

and $\int_M e^{-f}\, dV$ is constant. In particular, $\mathcal{F}(g_{ij}(t), f(t))$ is nondecreasing in time and the monotonicity is strict unless $g$ is a steady gradient soliton, i.e., $R_{ij} + \nabla_i \nabla_j f = 0$, which means that $g$ moves by the one-parameter group of diffeomorphisms generated by $\nabla F$. A similar monotonicity formula also holds for a related functional $\mathcal{W}(g_{ij}, f, \tau)$ depending on a further function $\tau$.

Perelman also introduced further quantities he called the entropy functional and the reduced volume and proved their monotonicity under the Ricci flow. These are more complicated to define than $\mathcal{F}$ and $\mathcal{W}$, but are more useful for local considerations, especially in studying the formation of singularities. Using these results, he proved a fundamental "no local collapsing" property. Namely, if $g_{ij}(t)$, $[0, T)$, $T < \infty$, is a solution of the Ricci flow on a compact $n$-manifold $M$, then there exist positive constants $\kappa$ and $\rho_0$ such that for any $t_0 \in [0, T)$ and any $x_0 \in M$, the solution $g_{ij}(t)$ is $\kappa$-noncollapsed at $(x_0, t_0)$ on all scales less than $\rho_0$. This means that for any $r \in (0, \rho_0)$, if $|\mathrm{Rm}(x, t)| \leq r^{-2}$ for all $x \in B_{t_0}(x_0, r)$ and $t \in [t_0 - r^2, t_0]$, then $\mathrm{Vol}_{t_0}(B_{t_0}(x_0, r)) \geq \kappa r^n$ (see Theorem 3.3.2).

A version of this result also holds on noncompact manifolds, provided the injectivity radius of the initial metric is uniformly bounded away from zero, and the metric at each time is complete with bounded curvature (see Theorem 3.4.2).

The injectivity radius estimate or "Little Loop Lemma" (Theorem 4.2.4) required by Hamilton follows from this and a result of Cheng, Li and Yau (see Theorem 4.2.2).

Perelman's second major contribution is a precise description of the singularity models $(N, g(t))$ that can arise by rescaling finite time singularities of the Ricci flow on a compact 3-manifold $M$. In particular, he showed that if $N$ is a complete, noncompact singularity model arising from a finite time singularity of the Ricci flow on a compact 3-manifold, then in a suitable scale, near infinity $N$ looks like a union of $\varepsilon$-necks, where an $\varepsilon$-neck is a region that is $\varepsilon$-close in the $C^{[\varepsilon^{-1}]}$ topology to $\mathbb{S}^2 \times [-\varepsilon^{-1}, \varepsilon^{-1}]$ with the usual metric. Moreover, $\varepsilon$ can be made arbitrarily small by going sufficiently far out to infinity in $N$, and this structure also holds on long time intervals in the past.

Since the singularity models arise by taking limits of rescalings of the solutions $g(t)$ near a singularity, one expects this structure to be reflected in the small scale geometry and topology of solutions $g(t)$ very close to a singularity. Somewhat more precisely, Perelman obtained a "canonical neighbourhood structure" near the singularity time $T$ for points $(x, t)$ where the curvature is sufficiently large. Namely, any such point has an open neighbourhood $\mathcal{N}$ such that (in a suitable scale) one of the following holds:

(i) $\mathcal{N}$ is a compact manifold without boundary with positive sectional curvature;

(ii) $\mathcal{N}$ is an evolving $\varepsilon$-neck;

(iii) $\mathcal{N}$ is an evolving $\varepsilon$-cap, which is a region diffeomorphic to the 3-ball $\mathbb{B}^3$ or to $\mathbb{RP}^3 \setminus \overline{\mathbb{B}}^3$ attached along its boundary to one end of an $\varepsilon$-neck.

Now define $\Omega$ to be the subset of $M$ where the curvature is bounded at time $T$. If $\Omega = \emptyset$, then the curvature blows up everywhere on $M$ as $t \to T$. In this case, either the manifold is compact and positively curved (therefore it is diffeomorphic to a spherical space form $\mathbb{S}^3/\Gamma$), or it is entirely covered by evolving $\varepsilon$-necks and evolving $\varepsilon$-caps shortly before the maximal time $T$. In the latter case it follows that $M$ is diffeomorphic to $\mathbb{S}^3$, $\mathbb{RP}^3$, $\mathbb{RP}^3 \# \mathbb{RP}^3$, or $\mathbb{S}^2 \times \mathbb{S}^1$. In each case $M$ satisfies the conclusion of the geometrization conjecture.

To describe the case $\Omega \neq \emptyset$ we need some terminology. An $\varepsilon$-horn is a metric on $\mathbb{S}^2 \times (-\varepsilon^{-1}, \varepsilon^{-1})$ which is approximately round on each $\mathbb{S}^2$ factor and such that the scalar curvature stays bounded on one end and goes to infinity on the other. A double $\varepsilon$-horn is defined similarly, with the scalar curvature going to infinity at both ends.

It can be shown that for a small positive constant $\rho$, the set $\Omega_\rho = \{x \in \Omega \colon R(x, T) \leq \rho^{-2}\}$ intersects only finitely components $\Omega_1, \dots, \Omega_k$ of $\Omega$, each such component has finitely many ends, and each end is an $\varepsilon$-horn. On the other hand, any component of $\Omega$ not intersecting $\Omega_\rho$ is either a capped $\varepsilon$-horn or a double $\varepsilon$-horn, and there may be infinitely many such components. By looking at the solution a short time before $T$, we see that the topology of $M$ can be recovered as follows. Take the components $\Omega_1, \dots, \Omega_k$, truncate their $\varepsilon$-horns, and glue to the boundary components a finite collection of tubes $\mathbb{S}^2 \times (-1, 1)$ and caps $\mathbb{B}^3$ or $\mathbb{RP}^3 \setminus \overline{\mathbb{B}}^3$. Thus $M$ is the connected sum of $\widetilde{\Omega}_1, \dots, \widetilde{\Omega}_k$, a finite number of copies of $\mathbb{S}^2 \times \mathbb{S}^1$, and a finite number of copies of $\mathbb{RP}^3$. Here each $\widetilde{\Omega}_j$ is obtained from $\Omega_j$ by truncating each $\varepsilon$-horn and gluing in a 3-ball, as just described. Thus to understand the topology of $M$ one only needs to understand the topologies of the $\widetilde{\Omega}_j$.

Once $M$ has been geometrically decomposed as above, the Ricci flow can be started again on each $\widetilde{\Omega}_j$ with the new initial metric that results after performing the surgery, and the whole procedure is repeated when a singularity next develops. At each surgery a small reduction in volume occurs, and this can be used to show that the surgery times are discrete. This relies on a rather delicate rescaling argument; successive surgeries must be performed with increasing accuracy.

This procedure essentially defines the Ricci flow with surgery, which exists either for all time, or else becomes extinct after finite time (when $\Omega_\rho = \emptyset$). It is clear from the above description of the surgery that if the manifold is geometrizable after a surgery, then it is also geometrizable just before that surgery. So if it can be shown that the manifold $M_t$ resulting from Ricci flow with surgery starting with $(M, g_0)$ is geometrizable for sufficiently large time, then the original manifold $M$ is also geometrizable.

In the case that the Ricci flow with surgery becomes extinct in finite time, $M$ is diffeomorphic to a connected sum of finitely many copies of $\mathbb{S}^2 \times \mathbb{S}^1$ and quotients $\mathbb{S}^3/\Gamma$. Perelman proposed a finite extinction time result for closed, orientable 3-manifolds whose prime decomposition (1) contains no $Y_k$ factors. A proof of this result was given by *T. Colding* and *W. Minicozzi* [J. Am. Math. Soc. 18, No. 3, 561–569 (2005; Zbl 1083.53058)]. The Poincaré conjecture then follows as a special case.

Longtime behaviour. To complete the proof of the geometrization conjecture one needs to understand the longtime behaviour of the Ricci flow with surgery. In 1999 Hamilton analyzed the behaviour of solutions of the normalized Ricci flow on a compact 3-manifold,

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \frac{2}{3} r g_{ij},$$

where $r = r(t)$ is the average of the scalar curvature, assuming a uniform curvature bound $|\mathrm{Rm}(x,t)| \leq C$ for all $t \geq 0$. He showed that the following two possibilities can occur:

(i) There is a sequence of times $t_k \to \infty$ and a sequence of diffeomorphisms $\varphi_k : M \to M$ such that the pullback metrics $\varphi_k^* g(t_k)$ converge to a metric of constant sectional curvature.

(ii) There is a finite collection (possibly empty) of complete hyperbolic manifolds $H_i$ of finite volume, and for each sufficiently large $t$, an embedding $\varphi_t \colon \coprod_i H_i \to M$ such that the pullback metrics $\varphi_t^* g(t)$ converge uniformly on any compact subset of $\coprod_i H_i$ to a metric of constant negative curvature. Furthermore, the tori in $H_i$ near infinity are mapped to incompressible tori in $M$. Finally, the complement of the image of $\varphi_t$ (the thin set) has bounded curvature and the injectivity radii go to zero as $t \to \infty$.

The structure of the thin set is completely understood, due to results of *J. Cheeger, K. Fukaya*, and *M. Gromov* [J. Am. Math. Soc. 5, No. 2, 327–372 (1992; Zbl 0758.53022)] on collapsing manifolds under a two-sided curvature bound: it is a graph manifold, and these have been topologically classified; after removing a finite collection of disjoint embedded tori, each component is an $\mathbb{S}^1$ bundle over a surface.

Perelman showed that this picture holds for longtime solutions of the Ricci flow, as well as for the Ricci flow with surgery, without requiring boundedness of the normalized curvature. Essentially, he proved curvature bounds that are sufficient to obtain the above conclusions. To prove that the thin set is a graph manifold he claimed a version of the collapsing result of Cheeger, Gromov and Fukaya assuming only a lower curvature bound; the proof of this has not appeared in the literature. A proof for closed manifolds was given by *T. Shioya* and *T. Yamaguchi* [Math. Ann. 333, No. 1, 131–155 (2005; Zbl 1087.53033)]; the authors use this in place of Perelman's claim. Using this weaker result, they prove the geometrization conjecture only for the cases not already covered by Thurston's theorem that Haken 3-manifolds (those containing an incompressible surface of genus at least one) are geometrizable.

The topology of longtime solutions of the Ricci flow on compact 3-manifolds is thus understood, and this completes the proof of the geometrization conjecture.

Concluding remarks. The authors have tried to make the exposition essentially self-contained and accessible to readers familiar with the basics of Riemannian geometry and elliptic and parabolic partial differential equations. Many of Hamilton's results are presented following his original papers very closely. Perelman's papers contain many key new ideas, but some of his proofs are a little sketchy. The authors have filled in the details, and in some cases they have modified Perelman's proofs.

Detailed expositions of the work of Perelman have also been presented in the notes of *B. Kleiner* and *J. Lott* ["Notes and commentary on Perelman's Ricci flow papers", available at `http://www.math.lsa.umich.edu/~lott/ricciflow/perelman.html`], and in the book by *J. Morgan* and *G. Tian* [Ricci flow and the Poincaré conjecture. Clay Mathematics Monographs 3. Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute (2007; Zbl 1179.57045)], in which they present a complete proof of the Poincaré conjecture (and more general results), but not of the geometrization conjecture.

There are also several useful surveys emphasizing different aspects of the theory. We mention in particular those of *W. P. Thurston* [Bull. Am. Math. Soc., New Ser. 6, 357–379 (1982; Zbl 0496.57005)], *P. Scott* [Bull. Lond. Math. Soc. 15, 401–487 (1983; Zbl 0561.57001)], *H.-D. Cao* and *B. Chow* [Bull. Am. Math. Soc., New Ser. 36, No. 1, 59–74 (1999; Zbl 0926.53016)], *J. W. Milnor* [Notices Am. Math. Soc. 50, No. 10, 1226–1233 (2003; Zbl 1168.57303)], *M. T. Anderson* [Notices Am. Math. Soc. 51, No. 2, 184–193 (2004; Zbl 1161.53350)], *J. W. Morgan* [Bull. Am. Math. Soc., New Ser. 42, No. 1, 57–78 (2005; Zbl 1100.57016)] and *T. Tao* ["Perelman's proof of the Poincaré conjecture: a nonlinear PDE perspective", preprint, `arxiv:math/0610903`]. The recent books of *B. Chow* and *D. Knopf* [The Ricci flow: an introduction. Mathematical Surveys and Monographs 110. Providence, RI: American Mathematical Society (AMS) (2004; Zbl 1086.53085)] and *B. Chow, P. Lu* and *L. Ni*, Hamilton's Ricci flow. Graduate Studies in Mathematics 77. Providence, RI: American Mathematical Society (AMS) (2006; Zbl 1118.53001)] are useful introductions to the Ricci flow.

In December 2006 the authors posted a revised version of the paper under review with the modified title "Hamilton-Perelman's proof of the Poincaré Conjecture and the Geometrization Conjecture" [`arxiv:`

math/0612069], to better reflect their view that the full credit of proving the Poincaré conjecture goes to Hamilton and Perelman. The references and attributions have also been updated and modified.

Reviewer: John Urbas (Canberra) (MR2233789)

**MSC:**

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|---|---|
| 53C44 | Geometric evolution equations (mean curvature flow, Ricci flow, etc.) (MSC2010) |
| 53C21 | Methods of global Riemannian geometry, including PDE methods; curvature restrictions |
| 57M40 | Characterizations of the Euclidean 3-space and the 3-sphere (MSC2010) |
| 57M50 | General geometric structures on low-dimensional manifolds |



**Keywords:**

Ricci flow; Ricci flow with surgery; Hamilton-Perelman theory; Poincaré conjecture; geometrization of 3-manifolds

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