Bourbaki, Nicolas

Elements of mathematics. Commutative algebra. Chapter 10. Reprint of the 1998 original. (Éléments de mathématique. Algèbre commutative. Chapitre 10.) (French) Zbl 1107.13002
Berlin: Springer (ISBN 3-540-34394-6/pbk). 191 p. (2007).

The tenth chapter of N. Bourbaki's book "Commutative Algebra" is a late child, and the youngest one of the Bourbaki dynasty, so far. The French original appeared as late as in 1998, that is 15 years after the publication of the last supplementary Chapters 8 and 9 (Zbl 0579.13001), but has never been reviewed ever since. Nevertheless, it represents a rather seamless continuation of the entirety of the foregoing, in part already quite classical chapters, this time emphasizing several advanced topics in commutative and homological algebra that are of fundamental importance in modern algebraic geometry.

The book under review is the faithful and unabridged reprint of the French original of Chapter 10 of N. Bourbaki's "Commutative Algebra" (Masson, Paris, 1998). As such it contains ten sections, each of which is subdivided into up to ten subsections, with the addition of a large amount of exercises, in traditional Bourbaki style, designed for each section separately.

Section 1 is devoted to the crucial notion of depth of a module and of an algebra, together with a systematic study of its basic algebraic and geometric properties. This section works up a great deal of a framework once introduced by A. Grothendieck in the 1960s through his treatises "Eléments de Géométrie Algébrique" (EGA) and his mimeographed exposes "Séminaire de Géométrie Algébrique" (SGA).

Section 2 provides the fundamentals of Cohen-Macaulay rings and modules, without being exhaustive, but with a special emphasis layed on finite algebras and flat algebras as ground rings.

Section 3 discusses the notions of projective dimension and injective dimension of a module, the homological dimension of a ring, their relation to the concept of depth, and Gorenstein rings.

Section 4 gives a brief introduction to regular rings and their basic (homological) properties, while Section 5 deals with complete intersections via completely secant ideals. This section also contains a characterization of graded regular rings as well as a description of the behaviour of complete intersections under base change.

Section 6 turns to base change properties of Cohen-Macaulay algebras, Gorenstein algebras, and regular algebras. At the end of this section, the class of so-called absolutely regular algebras is characterized in several different ways.

Section 7 treats the geometrically very fundamental concept of smooth algebras, including formally smooth algebras as topological rings, formally smooth quotients, and the Jacobi-Zariski criteria for smoothness.

The last three sections cover the fundamentals of Grothendieck duality theory in commutative and homological algebra, which has proved to be a crucial tool in both the algebraic and the geometric context.

Section 8 starts with duality for modules of finite length. This comprises the structure of indecomposable injective modules, Matlis duality, dualizing functors, Macaulay duality, and further related material. Section 9 focuses on dualizing modules, the existence oriteria for them, and on the related characterization of Gorenstein rings. The concluding Section 10 provides the technical rudiments of local cohomology and Grothendieck duality theory. This section is kept very concise and functional, with the focal point on Cohen-Macaulay base rings.

The last 30 pages of this volume are devoted to ten sets of exercises, each of which refers to one of the foregoing sections. Following Bourbaki's traditional style, most of these exercises are purely theoretical, extremely complex and directed to completing the respective material by further (advanced) theorems.

In this volume representing Chapter 10 of N. Bourbaki's "Commutative Algebra", and having been written another 15 years after the foregoing Chapters 8 and 9, another slight change of the overall disposition is quite unambiguous. There are barely any motivating or historical remarks in the course of the text, no references or hints for additional reading are given, and the authors solely concentrate on definitions, lemmas, propositions, theorems, and corollaries, with just a few examples given in between. On the other hand, the exposition is typically systematic, rigorous and elegant, at least so from the methodological and technical point of view.

In this regard, the present Chapter 10 of N. Bourbaki's "Commutative Algebra" must be seen as a highly important and valuable source book for seasoned mathematicians working in the fields of commutative algebra and algebraic geometry, although the geometric aspects of the treated algebraic material are not explicitely pointed out. At any rate, much of the existing folklore in this central area of commutative algebra has been profoundly revised, systematized, finally established, and made generally available for effective use in current research.

After all, this is exactly what Bourbaki stood and stands for, and therefore the present volume accurately proceeds with the group's original program initiated some 75 years ago, this time with reference to the current developments in modern commutative algebra and, implicitely, in algebraic geometry.

Reviewer: Werner Kleinert (Berlin)

MSC:

13-02	Research exposition (monographs, survey articles) pertaining to commutative algebra
13H10	Special types (Cohen-Macaulay, Gorenstein, Buchsbaum, etc.)
13C15	Dimension theory, depth, related commutative rings (catenary, etc.)
13C40	Linkage, complete intersections and determinantal ideals
13D05	Homological dimension and commutative rings
13D45	Local cohomology and commutative rings

Keywords:

research monographs; commutative rings and algebras; depth; Cohen-Macaulay rings; Gorenstein rings; regular rings; complete intersections; local cohomology; duality theory

Full Text: DOI