Sirokofskich, A.; **Dimitracopoulos, C.**
On a problem of J. Paris. (English) Zbl 1137.03032
J. Log. Comput. 17, No. 6, 1099-1107 (2007).

Let $\mathrm{I}\Delta_n$ denote the fragment of PA in which the induction axiom schema is restricted to Δ_n formulas, and $\mathrm{B}\Sigma_n$ denote the theory PA^- plus the least number axiom schema for Σ_n formulas. The paper concerns a problem set by J. Paris:

$$\textbf{(P)}\quad \text{Does } \mathrm{I}\Delta_n \text{ imply } \mathrm{B}\Sigma_n\ (n \geq 1)?$$

T. A. Slaman solved the problem in ["Σ_n-bounding and Δ_n-induction", Proc. Am. Math. Soc. 132, 2449–2456 (2004; Zbl 1053.03034)] for $n \neq 1$: $\mathrm{I}\Delta_n$ implies $\mathrm{B}\Sigma_n$ for $n > 1$. Indeed he proved $\mathrm{I}\Delta_n + \exp \Rightarrow \mathrm{B}\Sigma_n$ for all $n \geq 1$.

N. Thapen improved this result in ["A note on Δ_1 induction and Σ_1 collection", Fundam. Math. 186, 79–84 (2005; Zbl 1082.03050)]: $\mathrm{I}\Delta_1 + \mathrm{th}_p \Rightarrow \mathrm{B}\Sigma_1$ where th_p denotes the axiom $\forall x\, \exists y\, \exists z\, (x < p(y) \wedge z = x^y)$ in which p is any primitive recursive function.

Here the authors give alternative proofs for the above theorems of Slaman and Thapen, which enable one to isolate models of $\mathrm{I}\Delta_1$ in which $\mathrm{B}\Sigma_1$ could fail. This could serve, as the authors claim, toward a negative solution of the above problem **(P)** for $n = 1$, which is still open. It is also proved that if the answer to problem **(P)** for $n = 1$ is YES (i.e., that $\mathrm{I}\Delta_1 \Rightarrow \mathrm{B}\Sigma_1$) then there exists a countable model of $\mathrm{B}\Sigma_1$ which does not have any proper end extension to a model of $\mathrm{I}\Delta_1$.

Note that the problem of whether or not any countable model of $\mathrm{B}\Sigma_1$ have a proper end extension to a model of $\mathrm{I}\Delta_1$ is still open too.

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