Levine, Marc
The homotopy coniveau tower. (English) Zbl 1154.14005
J. Topol. 1, No. 1, 217-267 (2008).

The article under review is a major contribution to motivic homotopy theory, as it solves several of Voevodsky's open problems stated in [*V. Voevodsky*, Open problems in the motivic stable homotopy theory. I. Motives, polylogarithms and Hodge theory. Part I: Motives and polylogarithms. Papers from the International Press conference, Irvine, CA, USA, June 1998. Somerville, MA: International Press. Int. Press Lect. Ser. 3, No. I, 3–34 (2002; Zbl 1047.14012)]. As the author mentions, his original aim was to "give an alternative argument for the technical underpinnings of ... the construction of a spectral sequence from motivic cohomology to K-theory". The spin-off is an identification of Voevodsky's slice filtration, a proof of Voevodsky's connectivity conjecture, a computation of the slices of K-theory (which implies the desired spectral sequence), and an identification of the zero slice of the sphere spectrum with the Eilenberg-MacLane spectrum. All of this works over a perfect and sometimes infinite field k, with no restriction on the characteristic. Here are some details. For the purpose of this review, a homotopy invariant, Nisnevich-excisive presheaf $E\colon \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Spt}$ from the category of smooth k-schemes with values in the category of spectra will be called a *good theory*. The main example of a good theory is K-theory. Relying heavily on his work in [*M. Levine*, K-Theory 37, No. 1–2, 129–209 (2006; Zbl 1117.19003) and J. Algebr. Geom. 10, No. 2, 299–363 (2001; Zbl 1077.14509)], the author constructs, for any good theory E, the tower

$$\cdots \to E^{(p)} \to E^{(p-1)} \to \cdots \to E^{(0)} \sim E$$

of good theories mentioned in the title of the article under review. The spectrum $E^{(p)}(X)$ is the realization of a simplicial spectrum whose n-simplices are the homotopy colimit of homotopy fibers

$$\mathrm{hofib}\big(E(X \times \Delta^n) \to E(X \times \Delta^n - W)\big)$$

where W runs through the closed subsets of $X \times \Delta^n$ of codimension at least p which are in good position with respect to the faces. A localization theorem implies that the homotopy coniveau tower is compatible with taking $\mathbb{P}^1$-loops in the sense that there is a natural zig-zag of weak equivalences connecting $(\Omega_{\mathbb{P}^1} E)^{(p-1)}$ and $\Omega_{\mathbb{P}^1}\big(E^{(p)}\big)$. As a consequence, the good theory $E^{(0/1)}$ obtained as the homotopy fiber of $E^{(1)} \to E^{(0)}$ is birational and rationally invariant. Here are the applications. In the case of K-theory, the author identifies the layer $K^{(p/p+1)}(X)$ with the Eilenberg-MacLane spectrum corresponding to Bloch's cycle complex, which in turn produces the desired spectral sequence. Over an infinite perfect field, the homotopy coniveau tower is shown to coincide with Voevodsky's slice tower for S^1-spectra, thereby proving Voevodsky's connectivity conjecture in this case. In the case of $\mathbb{P}^1$-spectra, the comparison works over any perfect field. The author finally proves that the zero slice of the $\mathbb{P}^1$-sphere spectrum and the integral Eilenberg-MacLane spectrum are equivalent via a "reverse cycle map", whose construction is fairly involved. Voevodsky obtained the last result for fields of characteristic zero by a completely different method [*V. Voevodsky*, Proc. Steklov Inst. Math. 246, 93–102 (2004; Zbl 1182.14012)]. Modulo a conjecture which is proven by *P. Pelaez* [Multiplicative properties of the slice filtration, Thesis, `http://www.math.uiuc.edu/K-theory/0898/`], the last comparison supplies the slices of any $\mathbb{P}^1$-spectrum over a perfect field with a module structure over the integral Eilenberg-MacLane spectrum. If again the field has characteristic zero, or one uses the rational Eilenberg-MacLane spectrum instead, the slices of any $\mathbb{P}^1$-spectrum are thus motives [*O. Röndigs, P. A. Østvær*, C. R., Math., Acad. Sci. Paris 342, No. 10, 751–754 (2006; Zbl 1097.14016)] (see also [*O. Röndigs, P. A. Østvær*, Adv. Math. 219, 689–727 (2008; Zbl 1180.14015)]).

Reviewer: Oliver Röndigs (Osnabrück)

MSC:

14C25 Algebraic cycles
14F42 Motivic cohomology; motivic homotopy theory
19E08 K-theory of schemes
19E15 Algebraic cycles and motivic cohomology (K-theoretic aspects)
55P42 Stable homotopy theory, spectra

Keywords:
motivic homotopy theory; cycle complexes; slice filtration; spectral sequence; motivic cohomology; algebraic K-theory

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