Calta, Kariane; **Smillie, John**
Algebraically periodic translation surfaces. (English) Zbl 1151.57015
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A translation surface is a compact surface obtained by gluing a finite number of polygons in $\mathbb{R}^2$ along edges by parallel translations. Such a surface can also be defined as a closed Riemann surface equipped with a nonzero holomorphic one-form. There is a great variety of questions related to translation surfaces, that involve geometry, dynamics, number theory and moduli space theory (Teichmüller theory), and several works have been done on these questions during the last decades by various people.

In the paper under review, a new algebraic approach for studying these surfaces is developed, based on previous work of several authors. One of the starting points in this development is the natural action of the Lie group $\mathrm{SL}(2,\mathbb{R})$ on translation surfaces which gives rise to the Veech group $V(S)$ of a translation surface S, defined as the stabilizer of this surface in $\mathrm{SL}(2,\mathbb{R})$. If the Veech group is a lattice in $\mathrm{SL}(2,\mathbb{R})$, then the surface S is called a lattice surface. *R. Kenyon* and *J. Smillie*, in their paper [Comment. Math. Helv. 75, No. 1, 65–108 (2000; Zbl 0967.37019)], introduced a new invariant, which they called the J-invariant, and which takes its values in $\mathbb{R}^2 \wedge_{\mathbb{Q}} \mathbb{R}^2$, which they used as a tool in the classification of lattice surfaces that arise from acute rational triangular billiard tables. This invariant was later on used by Calta to develop algebraic criteria for the geodesic flow on a surface in a given direction to be periodic. It turns out that a projection of the J-invariant in a direction v is the Sah-Arnoux-Fathi (SAF) invariant associated to to an interval-exchange tranformation induced by the flow in the direction v. The authors, in the paper under review, study the set of directions in which the SAF-invariant vanishes. They show that these directions are described by a number field in $\mathbb{R}$ which they call the periodic direction field. They obtain several results connecting the existence of periodic directions and number fields. In particular, they study the J-invariant of a translation surface introduced by Kenyon and Smillie. They show that for every number field K there is a translation surface for which the periodic direction field is K. They study automorphism groups associated to a translation surface and relate them to the J-invariant. They relate the existence of decompositions of translation surfaces into squares with the total reality of the periodic direct ion field.

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MSC:

57M50 General geometric structures on low-dimensional manifolds
37D50 Hyperbolic systems with singularities (billiards, etc.) (MSC2010)
30F30 Differentials on Riemann surfaces
32G15 Moduli of Riemann surfaces, Teichmüller theory (complex-analytic aspects in several variables)
30F20 Classification theory of Riemann surfaces

Keywords:
translation surface; algebraic periodicity; J-invariant; SAF-invariant

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