Main purpose of the present article is to show the existence of a coarse moduli space of irreducible connections on vector bundles of rank 2 on \( \mathbb{P}^1(\mathbb{C}) \) having regular singularities in three fixed points, say, \( \{0, 1, \infty\} \). Put \( U_0 = \mathbb{P}^1(\mathbb{C}) - \{\infty\} = \mathbb{C} \) with coordinate \( z \) and \( U_\infty = \mathbb{P}^1(\mathbb{C}) - \{0\} = \mathbb{C} \) with coordinate \( x \) such that \( xz = 1 \). Let \( S = \{s_1, s_2, \ldots, s_n\} \) be a finite collection of points of \( X = \mathbb{P}^1(\mathbb{C}) \), \( n \geq 2 \), \( s_i \neq s_j \). If \( i \neq j \), \( z(s_i) = a_i \). Let \( \mathcal{E} \) be a holomorphic vector bundle of rank 2 on \( X = \mathbb{P}^1(\mathbb{C}) \). A connection \( \nabla \) on \( \mathcal{E} \) having regular singularities on \( S \) is, by definition: (1) \( \nabla \) is a holomorphic connection of \( \mathcal{E}_{|X-S} \) (i.e. a \( \mathbb{C} \)-linear map \( \nabla : \mathcal{E}_{|X-S} \to \Omega^1(\mathcal{E}_{|X-S}) \) satisfying \( \nabla(fe) = df \otimes e + f\nabla(e) \) for local sections \( f \) and \( e \) of \( \mathcal{O}_X \) and \( \mathcal{E} \), respectively); (2) for any \( s \in S \), there exist an open neighbourhood \( U \) of \( s \) in \( X \), a base \( \omega \) of \( \Omega^1_U \) and a base \( (e) \) of \( \mathcal{E}_U \) meromorphic in \( s \) (i.e. there exists an element \( T \in \text{GL}(2, \mathbb{C}) \Omega(X(S)) \) and a base \( (g) \) of \( \mathcal{E}_U \) with \( (e) = (g)T \) on \( T - \{s\} \) such that if we write \( \nabla(e) = \omega \otimes e \), then each component of the matrix \( M \) has a pole of order at most one at \( s \) and holomorphic on \( U - \{s\} \).

Put \( \nabla = \nabla(v), \frac{d}{dx} >, \) \( \nabla = \nabla(v), \frac{d}{dz} > \), where \( v \) is a local section of \( \mathcal{E} \). We have \( \nabla = -x^2 \nabla \) on \( U_0 \cap U_\infty \). The author first shows the following theorem, which enable him to reduce the moduli problem of connections to that of pairs of matrices:

**Theorem 1.2.1.** Let \( \mathcal{E} \) and \( S \) be the same as above and \( \nabla \) a connection on \( \mathcal{E} \) having regular singularities on \( S \). Then there exists a basis of \( \mathcal{E} \) meromorphic on \( S \) such that with respect to this basis, if \( S \subseteq U_0 \), then

\[
\nabla = (\frac{d}{dz}) + (A_1/(z-a_1)) + \cdots + (A_n/(z-a_n)),
\]

where \( A_1, \ldots, A_n \) are \( 2 \times 2 \) complex matrices and \( A_1 + \cdots + A_n = 0 \).

The author shows that the same theorem holds for algebraic vector bundles \( \mathcal{E} \) on \( \mathbb{P}^1(K) \) and an algebraic connection with regular singularities on \( S \), if \( K \) is an algebraically closed field of characteristic 0. He also shows that if \( K \) is not algebraically closed with \( \text{char}(K) = 0 \), then the theorem does not necessarily hold. (He gives the necessary and sufficient conditions that the theorem holds in this case: Theorem 1.3.6.)

Let \( P_m \) be the set consisting of pairs \((A_1, A_2)\) of \( K \)-valued \( 2 \times 2 \) matrices. The author defines two equivalence relations \( \sim \) on \( P_m \) by:

\[
(B_1, B_2) \sim (A_1, A_2) \text{ if } (B_1, B_2) = (T^{-1}A_1T, T^{-1}A_2T) \text{ for some } T \in \text{GL}(2, K);
\]

\[
(B_1, B_2) \approx (A_1, A_2) \text{ if } (B_1(z) + (B_2)/(z-1)) = T^{-1}((A_1)/(z) + (A_2)/(z-1))) + T^{-1}dT/dz \text{ for some } T \in \text{GL}(2, K[z, 1/z, 1/(z-1)]).
\]

Classifying elements of \( P_m \) relative to the first equivalence relation \( \sim \) agrees with classifying Fuchsian systems and classifying elements of \( P_m \) relative to the second equivalence relation \( \approx \) agrees with classifying connections (Theorem 1.2.1 and Lemma 3.1.5).

In § 2, the author classifies certain subsets of Fuchsian systems. Let \( V \) be the category of analytic spaces (or algebraic varieties). For any \( S \in V \), a family of pairs of matrices over \( S \) is a triple \((\mathcal{E}, A_1, A_2)\) consisting of a vector bundle \( \mathcal{E} \) of rank 2 on \( S \) and endomorphisms \( A_1, A_2 \) of \( \mathcal{E} \). Two families \((\mathcal{E}, A_1, A_2)\) and \((\mathcal{E}', A_1', A_2')\) on \( S \) are called equivalent, if there exist an open covering \( \{U_j\}_{j \in J} \) of \( S \) and isomorphisms \( \phi_j : \mathcal{E}|_{U_j} \to \mathcal{E}'|_{U_j} \) such that \( A_k|_{U_j} = \phi_j(A_k|_{U_j})\phi_j^{-1}, k = 1, 2, \) for all \( j \). We denote the equivalence class of \((\mathcal{E}, A_1, A_2)\) by \( c(\mathcal{E}, A_1, A_2) \). By \( F_m \) we mean a contravariant functor from \( V \) to \((\text{Set})\) determined by \( F_m(S) = \{c(\mathcal{E}, A_1, A_2)\}\).

Similarly one defines a subfunctor \( F \) of \( F_m \) by \( F(S) = \{c(\mathcal{E}, A_1, A_2)|((A_1(s), A_2(s))\text{ irreducible}, that is, } A_1(s) \text{ and } A_2(s) \text{have no common eigenvectors in } \mathcal{E}(s) \text{ for any } s \in S\}. \) The author shows that \( F_m \) has no coarse moduli space (proposition 2.2) but \( F \) has a coarse moduli space (Theorem 2.3.2). Here, a coarse moduli space for a functor \( G \) from \( V \) to \((\text{Set})\) is a pair \((N, \Phi)\) with \( N \subseteq V \), \( \Phi : G \to h_N = \text{Hom}(N, \cdot) \) such that \( \Phi(p) : G(p) \to h_N(p) \) is bijective where \( p \) is a point and for each \( L \subseteq V \) and each morphism \( \Psi : N \to h_L \), there is a unique morphism \( f : N \to L \) such that the diagram \( \Psi \Rightarrow \Phi^{-1} h_N, h_L \Rightarrow \Phi \) is commutative.

In § 3 the author classifies the irreducible connections on holomorphic vector bundles of rank 2 on \( \mathbb{P}^1(\mathbb{C}) \)
having regular singularities in three fixed points 0, 1, \infty. A family of irreducible connections on an analytic space \( S \) is, by definition, a pair \((\mathcal{E}, \nabla)\) where \(\mathcal{E}\) is a vector bundle of rank 2 on \(S \times \mathbb{P}^1(\mathbb{C})\) and \(\nabla\) is an irreducible relative connection on \(\mathcal{E}\) having regular singularities in \(\{0, 1, \infty\}\). ("Irreducible" means that the family of pairs \((A_1, A_2)\) on \(S\) associated with \(\nabla\) by Theorem 1.2.1 is irreducible.) Two pairs \((\mathcal{E}, \nabla)\) and \((\mathcal{E}', \nabla')\) are called equivalent if for each \(x \in X = S \times \mathbb{P}^1(\mathbb{C})\) there exist a neighbourhood \(U\) of \(x\) in \(X\) and an isomorphism \(\phi : E|_{U} \to E'|_{U}\) meromorphic along \(Y = S \times \{0, 1, \infty\}\) such that on \(U - Y\) we have \(z \nabla = z \nabla'\). Then the author shows that the functor \(G\) from \(V\) to \((\text{Set})\) defined by \(G(S) = \) the set of equivalence classes of \((\mathcal{E}, \nabla)\) has a coarse moduli space.

Reviewer: Kenji Ueno (Kyoto)

**MSC:**

- **14D20** Algebraic moduli problems, moduli of vector bundles
- **53C05** Connections (general theory)
- **32G13** Complex-analytic moduli problems
- **14D05** Structure of families (Picard-Lefschetz, monodromy, etc.)
- **14F06** Sheaves in algebraic geometry
- **14-02** Research exposition (monographs, survey articles) pertaining to algebraic geometry
- **53B15** Other connections

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Fuchsian system; coarse moduli space of irreducible connections on vector bundles of rank 2; regular singularities