Steele, G. Ander
Carmichael numbers in number rings. (English) Zbl 1176.11049
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A Carmichael number is a composite integer n such that $a^n \equiv a \pmod{n}$ for all integers a. It is known that there are infinitely many of them. Korselt's criterion states that a composite integer $n > 1$ is Carmichael iff n is squarefree and $p-1|n-1$ for all primes $p|n$.

The purpose of this paper is to characterise Carmichael ideals. Let K be an extension of $\mathbb{Q}$, and let I be a composite ideal in O_K, the ring of integers of K. The ideal I is Carmichael in O_K if for all $\alpha \in O_K$ we have

$$\alpha^{N(I)} \equiv \alpha \bmod I,$$

where $N(I) = \#O_K/I$. The author establishes a generalised Korselt's criterion. He also investigates when Carmichael numbers in the integers generate Carmichael ideals in the algebraic integers of an abelian field. Here he pays special attention to quadratic and cyclotomic fields. He also shows that if n is composite, then there exist infinitely many abelian number fields K such that $\gcd(n, \mathrm{Disc}(K)) = 1$ and n is not Carmichael in K. There are further results that we cannot mention for reasons of space.
The deepest ingredient in the proofs is a result due to *Y. Bugeaud, P. Corvaja* and *U. Zannier* [Math. Z. 243, No.1, 79–84 (2003; Zbl 1021.11001)] on an upper bound for the greatest common divisor of $a^n - 1$ and $b^n - 1$ and *D. R. Heath-Brown*'s [Q. J. Math., Oxf. II. Ser. 37, 27–38 (1986; Zbl 0586.10025)] classical result that all primes, with the possible exception of at most two, are primitive roots for infinitely many primes p.

Reviewer: Pieter Moree (Bonn)

MSC:

11R04 Algebraic numbers; rings of algebraic integers
11A07 Congruences; primitive roots; residue systems
11A41 Primes

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