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Finite descent obstructions and rational points on curves. (English) Zbl 1167.11024
Algebra Number Theory 1, No. 4, 349-391 (2007).

Let k be a number field, and X be a smooth projective k-variety. In this paper, the author considers a slightly modified set of adelic points

$$X(\mathbb{A}_k)_\bullet := \prod_{v\nmid\infty} X(k_v) \times \prod_{v|\infty} \pi_0\big(X(k_v)\big)$$

where the factors at infinite places v in the usual set of adelic points are reduced to the set of connected components of $X(k_v)$, and approaches the problem of determining the existence of a rational point on X using descent methods within $X(\mathbb{A}_k)_\bullet$ via *torsors under finite étale group schemes,* which can be viewed as generalizations of the n-Selmer group of an abelian variety. Before we discuss the main results, I would like to point out that the paper contains a nice overview of the descent via torsors under finite étale group schemes, which is an introduction suitable for those who are not familiar with this extent of a generalization of descent methods. One of the main results of this paper in the context of the Brauer-Manin obstruction is an improvement on *S. Siksek* [“The Brauer-Manin obstruction for curves having split Jacobians”, J. Théor. Nombres Bordx. 16, No. 3, 773–777 (2004; Zbl 1076.14033)], which is a positive answer toward the author’s conjecture that the Brauer-Manin obstruction is the only obstruction against rational points on curves – Skorobogatov first formulated it as a question. Using the descent via torsors under all finite abelian étale group schemes, the author establishes that *for all curves C* the subset $\mathcal{C}(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ consisting of points in $\mathcal{C}(\mathbb{A}_k)_\bullet$ which survive the descent is equal to the Brauer set $\mathcal{C}(\mathbb{A}_k)_\bullet^{\text{Br}}$, and proves the conjecture in the descent context, provided that there is a nonconstant k-morphism from C into an abelian variety A such that the divisible part of the Shafarevich-Tate group $\text{Ш}(k,A)$ of A is trivial and $A(k)$ is finite. The author believes that this condition is satisfied for all curves of genus ≥ 2. Let us review the descent via torsors under finite group schemes. An X-torsor under a finite étale group scheme G is a smooth projective variety Y with the following commutative diagram such that $Y\times G$ is identified with the fiber product $Y\times_X Y$:

$$\begin{array}{ccc} Y\times G & \xrightarrow{\mu} & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y] & \xrightarrow[\pi]{} & X \end{array}$$

where μ is a right action of G on Y, pr_1 is the projection, and π is a finite étale morphism. Note that to a point P in $X(k)$, we can associate an element in the Galois cohomology set $\mathrm{H}^1(k,G)$ as the group scheme G acts on the fiber $\pi\mathrm{Inv}(P)$. In other words, the X-torsor induces a map $\phi_Y : X(k) \to \mathrm{H}^1(k,G)$, and hence, the commutative diagram:

$$\begin{array}{ccc} X(k) & \xrightarrow{\phi_Y} & \mathrm{H}^1(k,G) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ X(\mathbb{A}_k)_\bullet & \xrightarrow[\delta]{} & \prod_v \mathrm{H}^1(k_v,G) \end{array}$$

whose setting is quite analogous to that of the n-Selmer group of an abelian variety. The set $\mathcal{C}ov(X)$ is defined to be the subset consisting of elements Q in $X(\mathbb{A}_k)_\bullet$ such that $\delta(Q)\in \mathrm{Img}(\mathrm{res})$ for all X-torsors under all finite étale group schemes, and the subsets $X(\mathbb{A}_k)_\bullet^{\text{f-sol}}$ and $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ are similarly defined with the solvable/abelian group schemes. Denoting by $\overline{X(k)}$ the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$, we have

$$X(k)\subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-sol}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet.$$

A significant part of this paper is for the development of a theory toward the occasions of equalities between these sets and toward the relationships with various Brauer sets, and as mentioned earlier, for the case of all curves, it is particularly fruitful.

For X being an abelian variety A, the author takes a more natural generalization of n-descent; namely,

$$0 \to \widehat{A(k)} \to \mathrm{Sel} \to T\ \text{Ш}(k, A) \to 0$$

where $\widehat{A(k)} = A(k) \otimes \hat{\mathbb{Z}}$, $\mathrm{Sel} = \lim \mathrm{Sel}^{(n)}$, and $T\,\text{Ш}(k, A)$ is the Tate module of $\text{Ш}(k, A)$. In this paper two main results for abelian varieties in this paper are introduced:

(1) For a set of finite places of k of good reduction for A and of density 1, we have canonical injective homomorphisms

$$\widehat{A(k)} \to \widehat{\mathrm{Sel}(k, A)} \to \prod_{v \in S} A(\mathbb{F}_v)$$

where the last map factors through $\prod_{v\in S} A(k_v)$, and $\widehat{A(k)} = \overline{A(k)}$ as images in $\prod_{v\in S} A(k_v)$.

This is an improvement on *J.-P. Serre* [Theorem 3, Sur les groupes de congruence des variétés abéliennes. II, Izv. Akad. Nauk SSSR, Ser. Mat. 35, 731–737 (1971; Zbl 0222.14025)];

(2) If $Z \subset A$ is a finite subscheme of A over k, then for a set S of places of k of density 1 the intersection $Z(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}(k, A)}$ in $\prod_{v\in S} A(k_v)/A(k_v)^0$ is the image of $Z(k)$.

The second result means that for a finite subscheme Z of A, the intersection $Z(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}(k, A)}$ is the only obstruction against rational points on Z.

The author also formulates an "Adelic Mordell-Lang Conjecture", and explains its implication on some subvarieties of A:

Adelic Mordell-Lang Conjecture: Let $X \subset A$ be a subvariety not containing a translate of a nontrivial subgroup of A. Then, there is a finite subscheme $Z \subset X$ such that $X(\mathbb{A}_k)_\bullet \cap \widehat{\mathrm{Sel}(k, A)} \subset Z(\mathrm{adel})_\bullet$.

This conjecture together with the second result for A implies that $X(k) = Z(k)$, and the chain of adelic subsets shown above collapses to $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. The author also remarks that the above conjecture is true when k is a global function field, A is ordinary, and X is not defined over k^p where p is the characteristic of k.

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MSC:

11G30 Curves of arbitrary genus or genus $\neq 1$ over global fields
14G05 Rational points
11G10 Abelian varieties of dimension > 1
14H30 Coverings of curves, fundamental group

Keywords:

rational points; descent obstruction; covering; twist; torsor under finite group scheme; Brauer-Manin obstruction

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