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Descent obstruction and étale Brauer-Manin obstruction. (Obstruction de descente et obstruction de Brauer-Manin étale.) (French) Zbl 1247.11090

Algebra Number Theory 3, No. 2, 237-254 (2009).

Given an algebraic variety X over a number field k with ring of adèles $\mathbb{A}_k$, one has the obvious inclusion of the set $X(k)$ of rational points into the set $X(\mathbb{A}_k)$ of adelic points, and the smaller set may be empty while the larger one is not, as classical counterexamples to the Hasse principle show. Several kinds of intermediate subsets provide general theories of obstruction to the Hasse principle. The first one, introduced by Manin and called the *Brauer-Manin obstruction*, concerns the subset $X(\mathbb{A}_k)^{\mathrm{Br}}$ of $X(\mathbb{A}_k)$ consisting of adelic points that are orthogonal to the Brauer group $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ with respect to the Brauer-Manin pairing. The *descent obstruction* is constructed by using torsors under linear algebraic groups: If $f : Y \xrightarrow{G} X$ is a torsor over X under a linear algebraic k-group G, every cocycle class $[\sigma] \in H^1(k, G)$ defines a twisted torsor $f^\sigma : Y^\sigma \to X$ over X. Adelic points coming from a twist of f (i.e. lying in $f^\sigma(Y^\sigma(\mathbb{A}_k))$ for some $[\sigma] \in H^1(k, G)$) form the descent subset $X(\mathbb{A}_k)^f$, and the intersection $X(\mathbb{A}_k)^{\mathrm{desc}}$ of all the descent subsets $X(\mathbb{A}_k)^f$ defined by torsors $f : Y \to X$ defines the descent obstruction. The *étale Brauer-Manin obstruction* is of some combinatorial nature. It is defined by the set

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} := \bigcap_{f:Y \xrightarrow{G} X,\, G \text{ finite}} \bigcup_{[\sigma]\in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}}).$$

An example constructed by *A. N. Skorobogatov* [Invent. Math. 135, No. 2, 399–424 (1999; Zbl 0951.14013)] shows that both the descent obstruction and the étale Brauer-Manin obstruction are strictly finer than the Brauer-Manin obstruction. More recently, *B. Poonen* [Ann. Math. (2) 171, No. 3, 2157–2169 (2010; Zbl 1284.11096)] has given examples of varieties X for which $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$ but $X(k) = \emptyset$, showing that the étale Brauer-Manin obstruction may not be the only obstruction to the Hasse principle.

Comparison between the descent and the étale Brauer-Manin obstructions has been discussed earlier by several authors. In the paper under review, the author proves the inclusion $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subseteq X(\mathbb{A}_k)^{\mathrm{desc}}$ for smooth projective geometrically integral varieties X over a number field k. This answers in the affirmative a question of Poonen [loc. cit.] as well as a similar question of *M. Stoll* [Algebra Number Theory 1, No. 4, 349–391 (2007; Zbl 1167.11024)], and together with the opposite inclusion proved by *A. Skorobogatov* [Math. Ann. 344, No. 3, 501–510 (2009; Zbl 1180.14017)], this implies that the descent obstruction coincides with the étale Brauer-Manin obstruction.

The main idea in the proof of the main theorem is to reduce the question of lifting adelic points for torsors under general groups to the special cases of torsors under finite or connected groups. This relies on a result about lifting 1-cococycles whose proof constitutes the most technical part.

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MSC:

11G35	Varieties over global fields
14G05	Rational points
11E72	Galois cohomology of linear algebraic groups
14G25	Global ground fields in algebraic geometry

Keywords:

Brauer-Manin obstruction; descent obstruction; Hasse principle; torsors; Galois cohomology

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