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Effectivization of a lower bound for $\|(4/3)^k\|$. (English. Russian original) Zbl 1230.11088

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Waring's problem is concerned with the representations of positive integers as sums of kth powers, i.e., with the solutions of the Diophantine equation

$$x_1^k + x_2^k + \cdots + x_s^k = N, \tag{1}$$

where N is a fixed positive integer and $x_1, \ldots, x_s$ are non-negative integer unknowns. This problem has two very different versions. First, one may ask what is the least s such that *all* $N \geq 1$ can be represented in the form (1). The least s with this property is usually denoted $g(k)$. Second, one may ask what is the least s such that all *sufficiently large* integers N can be represented in the above form; the least such s is usually denoted $G(k)$. The estimation of $G(k)$ is one of the central problems in additive number theory and has been a driving force behind the development of the circle method for the past ninety years. The value of $g(k)$, on the other hand, turns out to be determined by the arithmetic properties of certain relatively small N, and its study leads to some interesting questions on Diophantine approximation.

The paper under review establishes two inequalities related to the study of $g(k)$. The value of $g(k)$ depends on the inequality $\|(3/2)^k\| \geq (3/4)^k$, where $\|x\|$ denotes the distance from x to the nearest integer. Thus, several authors have given estimates of the form

$$\left\|(3/2)^k\right\| \geq C^k \qquad \text{for all integers } k \geq k_0,$$

with explicit values of C and k_0. In particular, the best result to date has been obtained by *V. Zudilin* [J. Théor. Nombres Bordx. 19, No. 1, 311–323 (2007; Zbl 1127.11049)], who gave such a bound with $C = 0.5803$. While the value of k_0 in Zudilin's work is effectively computable, it is not easy to compute it. In this paper, the author uses Zudilin's method to obtain a fully explicit, albeit slightly weaker, estimate:

$$\left\|(3/2)^k\right\| \geq (0.5795)^k \qquad \text{for all integers } k \geq 871,387,440,264.$$

He further shows that

$$\left\|(4/3)^k\right\| \geq (0.491)^k \qquad \text{for all integers } k \geq k_1,$$

where k_1 is an explicitly given number of the order of 5.868×10^{18}. This is also a fully explicit version of a result of Zudilin [op. cit.].

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