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The inverse problem in the calculus of variations is to determine when a given system of differential equations are the Euler-Lagrange equations for a variational problem defined by a Lagrangian L. If the Lagrangian L is invariant under the action of a Lie group G, then G is also a symmetry group of the Euler-Lagrange equations G. Thus the equivariant inverse problem arises, viz. if T is an invariant differential operator, then when is T the Euler-Lagrange operator of an invariant Lagrangian?

In this paper the equivariant inverse problem is solved for natural differential operators on Riemannian structures, i.e. operators $T = T[j^k g]$ which are defined upon the k-jet of a Riemannian metric and which satisfy the invariance condition $T[j^k[f^*(g)] = f^* T[j^k(g)]$ for all local diffeomorphisms f. It is shown that the obstructions to the construction of natural Lagrangians, $L = L[j^\ell(g)]$ for such natural operators T are generated by the secondary characteristic classes of Chern-Simon.

In particular, on even dimensional manifolds there are no obstructions while on 3 manifolds the so-called Cotton tensor $C^{ij} = \epsilon^{ihk} R^j_{h|k} + \epsilon^{jhk} R^i_{h|k}$, where R^j_h is the Ricci tensor, is the only natural tensor which is derivable from a variational principle but not an invariant variational principle.

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