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An adaptive wavelet method for solving high-dimensional elliptic PDEs. (English)

Zbl 1205.65313
Constr. Approx. 30, No. 3, 423-455 (2009).

Let $\Omega = (0,1)^n$, and let Γ_D be the union of one or more $(n-1)$-dimensional faces of $\partial\Omega$. For given $f \in (H^1_{0,\Gamma_D}(\Omega))'$, the authors study the numerical solution of the problem of finding $u \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$a(u,v) := \int_\Omega c_0 uv + \sum_{m=1}^n c_m \partial_m u\, \partial_m v = f(v), \quad v \in (H^1_{0,\Gamma_D}(\Omega))',$$

where $c_0 \geq 0$ and $c_m > 0$, $m = 1, \dots, n$, are constants.

The authors apply a tensor product basis $\{\psi_\lambda : \lambda \in \nabla\}$ constructed by a univariate $L^2(0,1)$-orthonormal piecewise polynomial wavelet basis. In this case, the condition number of the stiffness matrix $\kappa(A)$ is bounded uniformly in n, and c_0 and c_m, $m = 1, \dots, n$. Moreover, A is close to a sparse matrix.

The authors are interested in solutions u from the span of $\{\psi_\lambda : \lambda \in \Lambda_N\}$, where Λ_N is any subset with $\#\Lambda_N = N$. They give a detailed description of an adaptive wavelet algorithm for which the resulting approximations converge in energy norm with the same rate as the best approximations from the span of the best N tensor product wavelets. Moreover, the cost for producing these approximations will be proportional to their length with a constant factor that grows only linearly with N.

Reviewer: Gerlind Plonka-Hoch (Göttingen)

MSC:

65N30	Finite element, Rayleigh-Ritz and Galerkin methods for boundary value problems involving PDEs
46B28	Spaces of operators; tensor products; approximation properties
65T60	Numerical methods for wavelets
35J25	Boundary value problems for second-order elliptic equations
65F35	Numerical computation of matrix norms, conditioning, scaling
65Y20	Complexity and performance of numerical algorithms
65N12	Stability and convergence of numerical methods for boundary value problems involving PDEs

Keywords:

adaptive wavelet methods; best N-term approximations; tensor product approximation; sparse grids; matrix compression; optimal computational complexity; convergence; condition number; algorithm

Full Text: DOI

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