Elsenhans, Andreas-Stephan; Jahnel, Jörg

On the smallest point on a diagonal cubic surface. (English) Zbl 1233.11072

Exp. Math. 19, No. 2, 181-193 (2010).

The authors consider the height $m(S)$ of the smallest rational point (if there are such) and the Tamagawa-type number $\tau(S)$ that was defined in [*E. Peyre*, "Hauteurs et mesures de Tamagawa sur les variétés de Fano", Duke Math. J. 79, No. 1, 101–218 (1995; Zbl 0901.14025)] for a multitude of diagonal cubic surfaces S defined over $\mathbb{Q}$. It is known that such varieties do not satisfy the Hasse principle, as was shown first in [*J. W. S. Cassels* and *M. J. T. Guy*, "On the Hasse principle for cubic surfaces", Mathematika 13, Part 2, 111–120 (1966; Zbl 0151.03405)] for the equation

$$5x^3 + 9y^3 + 10z^3 + 12t^3 = 0.$$

A more systematic study was performed in the paper [*J.-L. Colliot-Thélène*, *D. Kanevsky* and *J.-J. Sansuc*, "Arithmétique des surfaces cubiques diagonales", Lect. Notes Math. 1290, 1–108 (1987Zbl 0639.14018)]. Conjectures about the growth of rational points of bounded height and other considerations suggest that $m(S)$ should be bounded by $\frac{C}{\tau(S)}$, where C is an absolute constant that does not depend on S. The authors test this hypothesis extensively in this paper for surfaces of the form

$$ax^3 + by^3 + 2z^3 + w^3 = 0,$$

where $1 \leq a \leq 3000$ and $1 \leq b \leq 300$, subject to some divisibility conditions such as a and b being odd and there existing an odd prime p dividing exactly one of a and b such that the positive p-adic valuation is not divisible by 3. This corresponds to the "first case" as described in the paper of Colliot-Thélène/Kanevski/Sansuc mentioned above, where the group $H^1(\mathbb{Q}, \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/3\mathbb{Z}$. Ignoring duplications and small values of a and b that are close enough so that there are rational points of uncharacteristically small height, they test a sample of 849,781 diagonal surfaces. Here is what they found. 802,891 of them have rational points, and they computed the quantities $m(S)$ and $\tau(S)$ mentioned above. A plot of these values reveals that the experiment agrees with expectations. However, the authors show that the expected inequality does not hold, in general. This leads them to ask whether it is true that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$, depending on ε, such that for every cubic surface,

$$m(S) < \frac{C(\varepsilon)}{\tau(S)^{1+\varepsilon}}.$$

The authors recall quickly the quantities mentioned above and then provide a very useful method (Theorem 4.5) to compute efficiently the action of the Galois group on $\operatorname{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{C}$, which is of dimension 7, and the values of Artin L-series at $s = 1$ for irreducible components of this representation. Then they explain in detail their algorithm for searching for rational points of small height, which is derived from their earlier work [*A.-S. Elsenhans* and *J. Jahnel*, "The asymptotics of points of bounded height on diagonal cubic and quartic threefolds", Lect. Notes Comput. Sci. 4076, 317–332 (2006; Zbl 1143.14300)] and on methods in the paper of Colliot-Thélène/Kanevsky/Sansuc mentioned above. Finally, they consider the case of diagonal cubic surfaces of the form

$$S^{(q)} : qx^3 + 4y^3 + 2z^3 + w^3 = 0$$

and show that for no constant C does the inequality (with notation as above)

$$m(S^{(q)}) < \frac{C}{\tau(S^{(q)})}$$

hold for all $q \in \mathbb{Z} \setminus \{0\}$.

Reviewer: Wayne Raskind (Tempe)

MSC:

11G35	Varieties over global fields
11G50	Heights
11G40	L-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture
14G05	Rational points
14G25	Global ground fields in algebraic geometry

Keywords:

diagonal cubic surface; Diophantine equation; smallest solution; naive height; E. Peyre's Tamagawa-type number

Full Text: DOI Euclid