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[For the entire collection see Zbl 0544.00006.]

A subset $\mathcal{S}$ of ${}^{\omega}\omega$ is said to be dominating [splitting] if for all $g \in^{\omega} \omega$ there is $f \in \mathcal{S}$ such that $g <^* f$ $[f \not<^* g]$. (The notation $f <^* g$ means $|\{n < \omega : f(n) \geq g(n)\}| < \omega$.) A subset $\mathcal{P}$ of $[\omega]^{\omega}$ is said to be splitting if for all $A \in [\omega]^{\omega}$ there exists $X \in \mathcal{P}$ such that $|A \cap X| = |A \setminus X| = \omega$. The symbols $\mathfrak{d}$, $\mathfrak{b}$ and $\mathfrak{s}$ denote, respectively, the minimal cardinalities of a dominating, an unbounded and a splitting set; and $\mathfrak{a}$ is the minimal cardinality of a maximal almost disjoint subset of $[\omega]^{\omega}$.

It is not difficult to prove in ZFC that $\aleph_1 \leq \mathfrak{s} \wedge \mathfrak{b} \leq \mathfrak{s} \vee \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{d} \vee \mathfrak{a} \leq \mathfrak{c} = 2^{\aleph_0}$, and $\mathfrak{b} \leq \mathfrak{a}$.

In the present article, extending results and responding to questions of E. van Douwen, B. Balcar, P. Simon, P. Nyikos, E. Miller and others, the author shows that each of the following is consistent with ZFC: (1) $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \mathfrak{a} = \mathfrak{c} = \aleph_2$; (2) $\aleph_1 = \mathfrak{s} < \mathfrak{b} = \mathfrak{c} = \aleph_2$; (3) $\aleph_1 < \mathfrak{s} = \mathfrak{b} = \mathfrak{c} = \aleph_2$; in each of the models constructed one has in addition $2^{\aleph_1} = \aleph_2$.

The required notions of forcing are described in considerable detail.

MSC:

03E35 Consistency and independence results

Keywords:

dominating sets; unbounded sets; splitting sets; minimal cardinalities; maximal almost disjoint subset; ZFC; forcing