Ebert, Johannes; Giansiracusa, Jeffrey

Pontrjagin-Thom maps and the homology of the moduli stack of stable curves. (English)

Zbl 1257.32011
Math. Ann. 349, No. 3, 543-575 (2011).

This paper is concerned with the study of the singular homology with field coefficients of the moduli stack $\overline{\mathcal{M}}_{g,n}$ of Deligne-Mumford-Kudsen stable curves of genus g with n labelled marked points. Particular emphasis is given to the study of the mod p homology rather then the rational homology, as the first has received little attention when compared with the second. As the mod p homology of $\overline{\mathcal{M}}_{g,n}$ and of its coarse moduli space are not necessarily isomorphic (as it is the case for the rational homology), the authors concentrate on the study of the homology of the stack rather then the space. The mod p cohomology of the substack $\mathcal{M}_{g,n}$ of smooth pointed curves was computed by *S. Galatius* in [Topology 43, No. 5, 1105–1132 (2004; Zbl 1074.57013)] in the Harer-Ivanov stable range, identifying a large number of torsion classes.

The main tool used in the paper is the Pontrjagin-Thom collapse map, in particular a generalized version of it holding for differentiable local quotient stacks; this generalization is discussed by the authors in Appendix A. This technique is then applied to the clutching (or gluing) morphisms given by identifying two marked points to form a node and whose images cover the irreducible components of the boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$. The outcome of this version of Pontrjagin-Thom's construction is the production of a map from "the homotopy type of the stack" $\overline{\mathcal{M}}_{g,n}$, which is a space with the same topological invariants as the stack, to a certain infinite loop space, whose homology is well known (and discussed in Appendix B of the paper). The main result of the paper states the surjectivity of those maps in certain stable ranges. Using this theorem the authors then use the injectivity of the pullback map to detect classes in $\overline{\mathcal{M}}_{g,n}$. In particular, large families of torsion classes, which are not reduction of rationally nontrivial classes nor coming from $\mathcal{M}_{g,n}$, are shown to exist in $\overline{\mathcal{M}}_{g,n}$.

Reviewer: Margarida Melo (Roma)

MSC:

14H15	Families, moduli of curves (analytic)
14D23	Stacks and moduli problems
14H10	Families, moduli of curves (algebraic)
55R40	Homology of classifying spaces and characteristic classes in algebraic topology

Full Text: DOI arXiv

References:

[1] Bers L.: Finite-dimensional Teichmüller spaces and generalizations. Bull. Am. Math. Soc. 5(2), 131–172 (1981) · Zbl 0485.30002 · doi:10.1090/S0273-0979-1981-14933-8

[2] Betley S.: Twisted homology of symmetric groups. Proc. Am. Math. Soc. 130(12), 3439–3445 (2002) · Zbl 1003.20046 · doi:10.1090/S0002-9939-02-06763-1

[3] Boldsen, S.K.: Improved homological stability for the mapping class group with integral or twisted coefficients, Ph.D. thesis, Århus Universitet, 2009, preprint, arXiv:0904.3269 · Zbl 1271.57052

[4] Bödigheimer, C.-F., Tillmann, U.: Stripping and Splitting Decorated Mapping Class groups, Cohomological Methods in Homotopy Theory (Bellaterra, 1998), Progr. Math., vol. 196, pp. 47–57. Birkhäuser, Basel (2001) · Zbl 0992.57014

[5] Deligne P., Mumford D.: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. 36, 75–109 (1969) · Zbl 0181.48803 · doi:10.1007/BF02684599

[6] Ebert, J.: The Homotopy Type of a Topological Stack. Preprint, arXiv:0901.3295

[7] Edidin, D.: Notes on the construction of the moduli space of curves, Recent Progress in Intersection Theory (Bologna, 1997), Trends Math., pp. 85–113. Birkhäuser Boston, Boston (2000) · Zbl 0990.14008

[8] Freed, D., Hopkins, M., Teleman, C.: Loop Groups and Twisted K-theory I. Preprint, arXiv:0711.1906, 2007

[9] Galatius S.: Mod p homology of the stable mapping class group. Topology 43(5), 1105–1132 (2004) · Zbl 1074.57013 ·

doi:10.1016/j.top.2004.01.011

[10] Galatius S., Eliashberg Y.: Homotopy theory of compactified moduli spaces. Oberwolfach Reports 13, 761–766 (2006)

[11] Galatius S., Madsen I., Tillmann U., Weiss M.: The homotopy type of the cobordism category. Acta Math. 202, 195–239 (2009) · Zbl 1221.57039 · doi:10.1007/s11511-009-0036-9

[12] Haefliger, A.: Groupoïdes d'holonomie et classifiant, Transversal structure of foliations, Toulouse (1982). Asterisque no. 116, pp. 70–97 (1984)

[13] Hanbury E.: Homological stability of non-orientable mapping class groups with marked points. Proc. Am. Math. Soc. 137(1), 385–392 (2009) · Zbl 1160.57015 · doi:10.1090/S0002-9939-08-09519-1

[14] Hatcher, A.: Vector Bundles and K-Theory, book in preparation, available at http://www.math.cornell.edu/\(\sim\)hatcher/VBKT/VBpage.html

[15] Heinloth, J.: Some notes on differentiable stacks, Mathematisches Institut, Seminars 2004/2005, Universität Göttingen, pp. 1–32 (2005) · Zbl 1098.14501

[16] Harris J., Morrison I.: Moduli of curves, Graduate Texts in Mathematics, vol. 187. Springer, New York (1998) · Zbl 0913.14005

[17] Hatcher, A., Wahl, N.: Stabilization for mapping class groups of 3-manifolds. Duke Math. J. (to appear) arXiv:math/0709.2173, 2007 · Zbl 1223.57004

[18] Knudsen F.: The projectivity of the moduli space of stable curves. II. The stacks M g,n . Math. Scand. 52(2), 161–199 (1983) · Zbl 0544.14020

[19] Kan D.M., Thurston W.P.: Every connected space has the homology of a K({\(\pi\)}, 1). Topology 15(3), 253–258 (1976) · Zbl 0355.55004 · doi:10.1016/0040-9383(76)90040-9

[20] Laumon, G., Moret-Bailly, L.: Champs algébriques, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 39. Springer, Berlin (2000) · Zbl 0945.14005

[21] May, J.P.: The Homology of E spaces, The homology of Iterated Loop Spaces, Lecture Notes in Mathematics, Vol. 533. Springer, Berlin (1976) · Zbl 0334.55009

[22] Milnor J.W., Moore J.C.: On the structure of Hopf algebras. Ann. Math. 81, 211–264 (1965) · Zbl 0163.28202 · doi:10.2307/1970615

[23] Moerdijk, I.: Orbifolds as groupoids: an introduction, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Am. Math. Soc., Providence, RI, pp. 205–222 (2002) · Zbl 1041.58009

[24] Mostow G.D.: Equivariant embeddings in Euclidean space. Ann. Math. 65, 432–446 (1957) · Zbl 0080.16701 · doi:10.2307/1970055

[25] McDuff D., Segal G.: Homology fibrations and the "group-completion" theorem. Invent. Math. 31(3), 279–284 (1975) · Zbl 0312.55021 · doi:10.1007/BF01403148

[26] Madsen I., Tillmann U.: The stable mapping class group and \({Q(\mathbb{C}P^\infty_+)}\) . Invent. Math. 145(3), 509–544 (2001) · Zbl 1050.55007 · doi:10.1007/PL00005807

[27] Madsen I., Weiss M.: The stable moduli space of Riemann surfaces: Mumford's conjecture. Ann. Math. 165, 843–941 (2007) · Zbl 1156.14021 · doi:10.4007/annals.2007.165.843

[28] Noohi, B.: Foundations of Topological Stacks I, preprint, arXiv:math/0503247, 2005

[29] Noohi, B.: Homotopy Types of Stacks. Preprint, arXiv:0808.3799, 2008 · Zbl 1149.22003

[30] Robbin J.W., Salamon D.A.: A construction of the Deligne–Mumford orbifold. J. Eur. Math. Soc. 8(4), 611–699 (2006) · Zbl 1105.32011

[31] Rudyak Y.B.: On Thom Spectra, Orientability, and Cobordism. Springer, Berlin (1998) · Zbl 0906.55001

[32] Segal G.: Configuration-spaces and iterated loop-spaces. Invent. Math. 21, 213–221 (1973) · Zbl 0267.55020 · doi:10.1007/BF01390197

[33] Segal G.: Categories and cohomology theories. Topology 13, 293–312 (1974) · Zbl 0284.55016 · doi:10.1016/0040-9383(74)90022-6

[34] Vakil, R.: The moduli space of curves and Gromov–Witten theory. In: Behrend, K., Manetti, M. (eds.) Enumerative Invariants in Algebraic Geometry and String Theory, Lecture Notes in Math., vol. 1947, pp. 143–198. Springer, Berlin (2008) · Zbl 1156.14043

[35] Zung N.T.: Proper groupoids and momentum maps: linearization, affinity, and convexity. Ann. Sci. École Norm. Sup. (4) 39, 841–869 (2006) · Zbl 1163.22001