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Topological modular forms [after Hopkins, Miller and Lurie]. (English) Zbl 1222.55003

Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011. Paris: Société Mathématique de France (SMF) (ISBN 978-2-85629-291-4/pbk). Astérisque 332, 221-255, Exp. No. 1005 (2010).

The paper has an expository character. The author describes the connection between homotopy theory and number theory. The results and notions discussed are put in historical context, which allows the reader to get a feeling of the development of the subject. The work of *D. Quillen* [Adv. Math. 7, 29–56 (1971; Zbl 0214.50502)] showed a strong connection between formal 1-parameter Lie groups and cohomology theories which admit natural Chern classes. The algebraic geometry of the formal Lie groups led to important conjectures in stable homotopy theory e.g., the nilpotence conjecture of Ravenel. The solution of this conjecture by *E. S. Devinatz, M. J. Hopkins* and *J. H. Smith* was one of the major achievements in homotopy theory of the 1980's [Ann. Math. (2) 128, No. 2, 207–241 (1988; Zbl 0673.55008) and ibid. 148, No.1, 1–49 (1998; Zbl 0927.55015)].

The 1-parameter formal Lie groups come from algebraic groups of dimension 1. Among these, elliptic curves as they can be considered in families over the base scheme S play an important role. The concept of elliptic cohomology provided an input, from homotopy theory, for a new field, namely the derived algebraic geometry, to emerge. The derived algebraic geometry also has its origins in various other branches of mathematics: in geometry [cf. the work of *J.-P. Serre* and *P. Gabriel*, Algébre locale, multiplicités. Paris: Cours professé au Collége de France. (1960; Zbl 0091.03701)] and [*L. Illusie*, Complexe cotangent et déformations. I. (The cotangent complex and deformations. I.). Lecture Notes in Mathematics. 239. Berlin-Heidelberg-New York: Springer-Verlag. (1971; Zbl 0224.13014)], in algebraic K-theory [cf. *J. F. Jardine*, Can. J. Math. 39, No. 3, 733–747 (1987; Zbl 0645.18006)].

The author describes and explains the theorem of Hopkins and Miller refined by Jacob Lurie which says that the compactified Deligne-Mumford moduli stack of elliptic curves is canonically and essentially uniquely an object in derived algebraic geometry. The homotopy global sections of this derived stack form the ring spectrum of topological forms. The author also describes the impact topological modular forms have in the theory of Witten genus, in the theory of modular forms and in understanding homotopy groups of spheres. The paper is written very clearly with exact references.

For the entire collection see [Zbl 1192.00071].

Reviewer: Piotr Krasoń (Szczecin)

MSC:

55N34 Elliptic cohomology
11F23 Relations with algebraic geometry and topology
14D23 Stacks and moduli problems
14H52 Elliptic curves
55N22 Bordism and cobordism theories and formal group laws in algebraic topology

Keywords:

derived algebraic geometry; elliptic cohomology; Deligne-Mumford moduli stack; E_∞-spectra; modular forms; topological modular forms