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Existence of Chebyshev centers in spaces of vector-valued continuous functions. (English) Zbl 0591.41022

In this paper we study the existence of Chebyshev centers in spaces of vector-valued continuous functions. More precisely, given a topological space $X$ and a Banach space $E$, consider $C_b(X; E)$, the space of all bounded continuous functions from $X$ into $E$ endowed with the sup-norm $\|f\| = \sup\{\|f(x)\|; x \in X\}$. When $W \subset C_b(X; E)$ is a closed vector subspace one asks whether any bounded subset $B$ of $C_b(X; E)$ has a Chebyshev center with respect to $W$, i.e., when there exists $g \in W$ such that $\sup_{f \in B} \|g - f\| \leq \inf_{h \in W} \sup_{f \in B} \|f - h\| = \text{rad}(B; W)$ (the number $\text{rad}(B; W)$ is called the Chebyshev radius of $B$ with respect to $W$).

To answer this question we introduce the following definition: Let $\mathcal{V}$ be a collection of closed vector subspaces of a Banach space $E$. We say that $\mathcal{V}$ has property (P) if, for every $\epsilon > 0$ and $R > 0$, there exists a $\delta > 0$ such that, given $V \in \mathcal{V}$ and a precompact subset $K \subset B(0, R + \delta)$ with $\text{rad}(K; V) \leq R$, there exists $v \in V$ such that $\|v\| \leq \epsilon$ and $K \subset \overline{B}(v, R)$.

An example of the results obtained in this article is the following theorem: Let $X$ be a paracompact Hausdorff space and let $E$ be a Banach space. Let $W \subset C_b(X; E)$ be a closed vector subspace such that:
(1) the family $\mathcal{V} = \{W(x); x \in X\}$ has property (P), where $W(x)$ is the closure in $E$ of $\{ g(x); g \in W \}$.
(2) for every $g \in C_b(X; E)$, if $g(x) \in W(x)$ for all $x \in X$, then $g \in W$. Then $W$ has the Chebyshev center property with respect to the family of all non-empty precompact subsets of $C_b(X; E)$.

MSC:
41A65 Abstract approximation theory (approximation in normed linear spaces and other abstract spaces)
41A50 Best approximation, Chebyshev systems

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