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On deviations, defects and asymptotic functions of meromorphic functions. (English)

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Let f be a meromorphic function and denote by $\pi A(r,f)$ the spherical area (area on the Riemann sphere) of $f(\{z:|z|\le r\})$. Set $\mathcal{L}(r,\infty,f)=\max_{|z|=r}\log^+|f(z)|$ and $\mathcal{L}(r,a,f)=\mathcal{L}(r,\infty,1/(f-a))$, where $a\in\mathbb{C}$.

In 1994, *W. Bergweiler* and *H. Bock* proved [J. Anal. Math. 64, 327–336 (1994; Zbl 0828.30013)] that for a meromorphic function of infinite lower order,

$$\liminf_{r\to\infty}\frac{\mathcal{L}(r,\infty,f)}{rT'_-(r,f)}\le\pi,$$

where $T'_-(r,f)$ is the left derivative of the Nevanlinna characteristic function. In connection with this result, *A. Eremenko* [Complex Variables, Theory Appl. 34, No. 1–2, 83–97 (1997; Zbl 0905.30025)] introduced the quantity

$$b(a,f)=\liminf_{r\to\infty}\frac{\mathcal{L}(r,\infty,f)}{A(r,f)}.$$

Since $rT'_-(r,f)=A(r,f)+O(1)$ $(r\to\infty)$, the theorem of Bergweiler and Bock implies that $b(a,f)\le\pi$ for each $a\in\mathbb{C}$. In [Eremenko, loc. cit.] an analogue of the deficiency relation for magnitudes of deviations $b(a,f)$ was obtained. A counterpart for the unit disk was introduced in [the second author and *I. G. Nikolenko*, Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky 2002, No. 2, 25–28 (2002; Zbl 1074.30519)]:

Theorem K. For a meromorphic function f such that the set $\{a\in\overline{\mathbb{C}}:b(a,f)>0\}$ contains more than one point we have

$$\sum_{a\in\mathbb{C}}b(a,f)\le 2\pi.$$

In a series of works, e.g. [the second author, Sb. Math. 189, No. 6, 875–899 (1998); translation from Mat. Sb. 189, No. 6, 59–84 (1998; Zbl 0942.30019); the authors, Math. Notes 85, No. 1, 20–33 (2009); translation from Mat. Zametki 85, No. 1, 22–35 (2009; Zbl 1177.30033)], the authors have systematically studied properties of the quantity $b(a,f)$.

In particular, they proved the following theorem:

Theorem M. Let f be an entire function of lower order $\lambda>0$ and denote by M the set of all rational functions. Then the set $\Omega=\{q\in M:b(q,f)>0\}$ is at most countable and

$$\sum_{q\in M}b(a,f)\le\begin{cases}\frac{\pi}{\sin\pi\lambda} & \text{if } 0<\lambda\le 0.5,\\ \pi & \text{if } 0.5<\lambda\le\infty.\end{cases}$$

In the present paper the authors try to extend the results to meromorphic functions.

Theorem 1. Let f be a meromorphic function of lower order λ, $0<\lambda\le\infty$, with $N(r,\infty,f)=o(T(r,f))$ $(r\to\infty)$, and denote by M the set of all rational functions. Then the set $\Omega=\{q\in M:b(q,f)>0\}$ is at most countable and

$$\sum_{q\in M}b(q,f)\le\begin{cases}\frac{\pi}{\sin\pi\lambda} & \text{if } 0<\lambda\le 0.5,\\ \pi & \text{if } 0.5<\lambda\le\infty.\end{cases}$$

Theorem 2. Let f be a meromorphic function of lower order λ, $0<\lambda\le\infty$, let also P_d be a set of all polynomials of degree at most d. Then

$$\sum_{q\in P_d}b(q,f)\le\begin{cases}\pi(d+2)\sqrt{\Delta(2-\Delta)} & \text{if } \lambda\ge 0.5 \text{ or } 0<\lambda<0.5 \text{ and } \sin\frac{\pi\lambda}{2}\ge\frac{\Delta}{2},\\ \pi(d+2)(\Delta\,ctg(\pi\lambda)+tg\,\frac{\pi\lambda}{2}) & \text{if } 0<\lambda<0.5 \text{ and } \sin\frac{\pi\lambda}{2}<\frac{\Delta}{2},\end{cases}$$

where $\Delta = \Delta(0, f^{(d+1)})$ is the Valiron deficiency of $f^{(d+1)}$ at the origin.

As a consequence, results on asymptotic polynomials are obtained.

The proofs are based on a combination of Petrenko's method and techniques of Bearnstein's $*$-function and Pólya peaks for functions of infinite lower order introduced by Bergweiler and Bock.

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30D30 Meromorphic functions of one complex variable, general theory

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