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Advanced topics in linear algebra. Weaving matrix problems through the Weyr Form. (English) Zbl 1235.15013

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The Weyr form will be unfamiliar to most mathematicians. As the authors explain, the form was introduced by the Austria-Hungarian mathematician Eduard Weyr (1852–1903) at roughly the same time as the Jordan form. It can be described as follows. Let N be an $n \times n$ nilpotent matrix of nilpotent index r, say, over an algebraically closed field F. The familiar Jordan form of N arises by choosing a basis of the underlying space of the form $U_1, \dots, U_s$ where each sublist U_i is a basis of an N-cyclic subspace of dimension m_i, say, and $r = m_1 \geq \cdots \geq m_s$. On the other hand the Weyr form for N arises by choosing a basis $V_1, \dots, V_r$ for the underlying space such that V_1 is a basis for the null space of N, and NV_i is an initial sublist of V_{i-1} for $i > 1$. If $n_i := |V_i|$ then $n_1 \geq \cdots \geq n_r$, and $(n_1, \dots, n_r)$ and $(m_1, \dots, m_s)$ are conjugate partitions of n. More generally if T is an arbitrary $n \times n$ complex matrix with eigenvalues $\lambda_1, \dots, \lambda_t$, then a basis for its Weyr form is the concatenation of bases for the Weyr forms of the $T - \lambda_i I$ restricted to the generalized λ_i-eigenspace $(i = 1, \dots, t)$. Part I of this book is an elementary introduction to the Weyr form, its elementary properties, and advocacy for the benefits of this form vis-à-vis the Jordan form. Part I ends with a chapter introducing modules, von Neumann regularity, and a description of the Weyr form in the setting of von Neumann regular rings.

However, for some readers, Part II of the book will be more interesting, and it is a pity that the title does not reflect this. The theme in Part II is the study of the commutative subalgebras of the full matrix algebra $M_n(F)$. A classical theorem of Schur states that the largest F-dimension of a commutative subalgebra of $M_n(F)$ is $\lfloor n^2/4 \rfloor + 1$. The question arises as to what is the largest F-dimension of a k-generator commutative subalgebra. Trivially this is n if $k = 1$ and in 1961 *M. Gerstenhaber* [Ann. Math. (2) 73, 324–348 (1961; Zbl 0168.28201)] showed that the same bound also holds for $k = 2$. In Chapter 5 the authors give a proof of this theorem (using the Weyr form) and related theorems, including some special results about the case $k = 3$. It is an open question as to whether every 3-generator commutative subalgebra has dimension $\leq n$.

In Chapter 6 the concept of "approximately simultaneously diagonalizable" (ASD) is introduced. Matrices $A_1, \dots, A_k$ in $M_n(\mathbb{C})$ are ASD if for a standard norm $\| \ \|$ it is true that for each $\varepsilon > 0$ there exist diagonalizable $B_1, \dots, B_k$ such that $\|A_i - B_i\| < \varepsilon$ for each i. In such a case, the A_i are necessarily pairwise commutative. In 1955 *T. S. Motzkin* and *O. Taussky*, [Trans. Am. Math. Soc. 80, 387–401 (1955; Zbl 0067.25401)] showed that conversely any pair of commuting matrices have the ASD property. However, when $k \geq 4$ and $n \geq 4$ commutativity does not imply ASD. The authors include two proofs of the Motzkin-Taussky theorem and show that it implies Gerstenhaber's theorem when $F = \mathbb{C}$. They then look at some special cases where the corresponding theorem for $k = 3$ has been proved. It is known that commutativity does not imply ASD when $k = 3$ and $n > 28$ (see below), but recent work by several authors has shown that commutativity implies ASD when $k = 3$ and $n \leq 8$.

Chapter 7 gives an elementary introduction to algebraic subvarieties of the affine variety $\mathbb{A}^n$ and then considers the variety $\mathcal{C}(k, n) \subseteq \mathbb{A}^{kn^2}$ consisting of the commuting k-tuples $(A_1, \dots, A_k)$ in $M_n(\mathbb{C})$. It is shown that commutativity of k-tuples implies the ASD property exactly when the variety $\mathcal{C}(k, n)$ is irreducible. Motzkin and Taussky proved that $\mathcal{C}(2, n)$ is irreducible (this gives another proof of the Motzkin-Taussky theorem above). Moreover, $\mathcal{C}(k, n)$ is irreducible for all k when $n \leq 3$ and is reducible when $k \geq 4$ and $n \geq 4$, so it remains to determine when $\mathcal{C}(3, n)$ is irreducible. The authors give a exposition of a version of *R. M. Guralnick*'s theorem [Linear Multilinear Algebra 31, No. 1–4, 71–75 (1992; Zbl 0754.15011)] which shows that $\mathcal{C}(3, n)$ is reducible for $n \geq 29$, and related results.

The authors suggest that the book is suitable for an undergraduate course, and have taken trouble to include careful details in their arguments. The topics covered by the book appear to be a rather uneasy compromise between the three authors' interests; some chapters are very leisurely and others are quite specialized. The second chapter gives a clear introduction to the Weyr form, but the idea of using the Weyr form as a unifying theme appears weak to the reviewer. On the other hand, Part II of the book is a good exposition of interesting material on matrix commutativity and of the underlying algebraic geometry, and it may be hoped that the book will introduce new readers to this still developing area.

Reviewer: John D. Dixon (Ottawa)

MSC:

15A21	Canonical forms, reductions, classification	Cited in **19** Documents
15-02	Research exposition (monographs, survey articles) pertaining to linear algebra	
15A30	Algebraic systems of matrices	
15A27	Commutativity of matrices	

Keywords:

Weyr form; commutative matrices; diagonalizable matrix; approximately simultaneously diagonalizable; Jordan form; irreducible; von Neumann regular rings