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Nonnegatively curved fixed point homogeneous 5-manifolds. (English) Zbl 1239.53046

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Let (M, g) be a closed Riemannian manifold endowed with an effective smooth action by a compact Lie group G. If the action has fixed points, then $\dim(M/G)$ is bounded below by the dimension of the fixed point set and one defines the fixed point cohomogeneity to be

$$\operatorname{cohomfix}(M, G) := \dim(M/G) - \dim(\operatorname{Fix}(M, g)) - 1 \geq 0\,.$$

If the fixed point cohomogeneity of the action is 0, the action is said to be *fixed point homogeneous* and (M, g) is said to be a *fixed point homogeneous manifold*; in this setting, the fixed point set has codimension 1 in the orbit space. The authors show:

Main Theorem: Let M^5 be a closed, simply connected, 5-dimensional nonnegatively curved fixed point homogeneous G-manifold. Then G is one of the groups $\{SO(5), SO(4), SU(2), SO(3), S^1\}$ and one has the following classification:

(a) If $G \in \{SO(5), SO(4), SU(2)\}$, then M is diffeomorphic to S^5.

(b) If $G \in \{SO(3), S^1\}$, then M is diffeomorphic to S^5 or to one of the two bundles over S^2 with fiber S^3.

The authors note that the list of fixed point homogeneous 5-manifolds in the main theorem contains every known closed simply connected 5-manifold of nonnegative sectional curvature except for the Wu manifold $SU(3)/SO(3)$. Section 1 of the paper contains an introduction to the subject at hand. In Section 2, basic facts about group actions and Alexandrov spaces are recalled. Section 3 contains the proof of the main theorem; the cases $\{SO(5), SO(4), SU(2), SO(3)\}$ are treated using standard classification results. The case $G = S^1$ has to be treated separately; the hypothesis of nonnegative curvature enables the authors to show by looking at the orbit space structure that M^5 decomposes as the union of two disk bundles over smooth submanifolds of M^5 one of which is a 3-dimensional component of the fixed point set; after examining $H_2(M^5; \mathbb{Z})$, the conclusion follows from the Barden-Smale classification of smooth closed simply connected 5-manifolds [*D. Barden*, Ann. Math. (2) 82, 365–385 (1965; Zbl 0136.20602); *S. Smale*, Ann. Math. (2) 75, 38–46 (1962; Zbl 0101.16103)].

Reviewer: Peter B. Gilkey (Eugene)

MSC:

53C20 Global Riemannian geometry, including pinching
57S15 Compact Lie groups of differentiable transformations

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References:

[1] Barden, D., Simply connected five-manifolds, Ann. Math., 82, 365-385, (1965) · Zbl 0136.20602

[2] Berard-Bergery, L., LES variétés riemanniennes homogènes simplement connexes de dimension impaire á courboure strictement positive, J. Math. Pures Appl., 55, 47-67, (1976) · Zbl 0289.53037

[3] Bredon G.E.: Introduction to Compact Transformation Groups, Pure and Applied Mathematics, vol. 46. Academic Press, New York, London (1972) · Zbl 0246.57017

[4] Burago D., Yuri B., Ivanov S.: A Course in Metric Geometry, Graduate Studies in Mathematics, vol. 33. American Mathe-

matical Society, Providence, RI (2001) · Zbl 0981.51016

[5] Cheeger, J.; Gromoll, D., On the structure of complete manifolds of nonnegative curvature, Ann. Math., 96, 413-443, (1972) · Zbl 0246.53049

[6] Galaz-Garcia, F.: Nonnegatively curved fixed point homogeneous manifolds in low dimensions. Geom. Dedicata, to appear. arXiv:0911.1254v1 [math.DG] · Zbl 1254.53059

[7] Galaz-Garcia, F., Searle, C.: Nonnegatively curved 5-manifolds with almost maximal symmetry rank. Preprint (2011) (see arXiv:0906.3870v1 [math.DG]) · Zbl 0929.53017

[8] Grove, K.: Critical point theory for distance functions. Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), Proceedings of the Symposium in Pure Mathematics, vol. 54, Part 3, pp. 357-385. American Mathematical Society, Providence, RI (1993) · Zbl 0806.53043

[9] Grove, K.: Geometry of, and via, symmetries. Conformal, Riemannian and Lagrangian geometry (Knoxville, TN, 2000), University Lecture Series, vol. 27, pp. 31-53. American Mathematical Society, Providence, RI (2002)

[10] Grove, K.; Searle, C., Positively curved manifolds with maximal symmetry-rank, J. Pure Appl. Algebr., 91, 137-142, (1994) · Zbl 0793.53040

[11] Grove, K.; Searle, C., Differential topological restrictions curvature and symmetry, J. Differ. Geom., 47, 530-559, (1997) · Zbl 0929.53017

[12] Grove, K.; Ziller, W., Curvature and symmetry of Milnor spheres, Ann. Math., 152, 331-367, (2000) · Zbl 0991.53016

[13] Grove, K.; Wilking, B.; Ziller, W., Positively curved cohomogeneity one manifolds and 3-Sasakian geometry, J. Differ. Geom., 78, 33-111, (2008) · Zbl 1145.53023

[14] Hamilton, R.S., Four-manifolds with positive curvature operator, J. Differ. Geom., 24, 153-179, (1986) · Zbl 0628.53042

[15] Hoelscher, C., Classification of cohomogeneity one manifolds in low dimensions, Pac. J. Math., 246, 129-185, (2010) · Zbl 1204.53029

[16] Kollár, J., Circle actions on simply connected 5-manifolds, Topology, 45, 643-671, (2006) · Zbl 1094.57020

[17] Orlik, P., Raymond, F.: Actions of \textit{SO}(2) on 3-manifolds, Proceedings of the Conference on Transformation Groups (New Orleans, La., 1967), pp. 297-318. Springer, New York (1968) · Zbl 0289.53037

[18] Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. Preprint (2002), arXiv:math/0211159v1 [math.DG]. · Zbl 1130.53001

[19] Perelman, G.: Ricci flow with surgery on three-manifolds. Preprint (2003), arXiv:math/0303109v1 [math.DG] · Zbl 1130.53002

[20] Raymond, F., Classification of the actions of the circle on 3-manifolds, Trans. Am. Math. Soc., 131, 51-78, (1968) · Zbl 0157.30602

[21] Searle, C., Cohomogeneity and positive curvature in low dimensions, Math. Z., 214, 491-498, (1993) · Zbl 0804.53057

[22] Searle, C., Corrigendum, Math. Z., 214, 491-498, (1993) · Zbl 0804.53057

[23] Searle, C.; Yang, D., On the topology of nonnegatively curved simply-connected 4-manifolds with continuous symmetry, Duke Math. J., 74, 547-556, (1994) · Zbl 0824.53036

[24] Smale, S., On the structure of 5-manifolds, Ann. Math., 75, 38-46, (1962) · Zbl 0101.16103

[25] Verdiani, L., Cohomogeneity one Riemannian manifolds of even dimension with strictly positive sectional curvature, I, Math. Z., 241, 329-339, (2002) · Zbl 1028.53035

[26] Verdiani, L., Cohomogeneity one manifolds of even dimension with strictly positive sectional curvature, J. Differ. Geom., 68, 31-72, (2004) · Zbl 1100.53033

[27] Wallach, N., Compact homogeneous Riemannian manifolds with strictly positive curvature, Ann. Math., 96, 277-295, (1972) · Zbl 0261.53033

[28] Wilking, B., Positively curved manifolds with symmetry, Ann. Math., 163, 607-668, (2006) · Zbl 1104.53030

[29] Wilking, B.: Nonnegatively and positively curved manifolds. Surveys in differential geometry. vol. XI, Surveys in differential Geometry, vol. 11, pp. 25-62. International Press, Somerville, MA (2007) · Zbl 1162.53026