Haberl, Christoph; **Schuster, Franz E.**; **Xiao, Jie**

An asymmetric affine Pólya-Szegő principle. (English) Zbl 1241.26014

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The classical Pólya-Szegő principle states that the symmetric rearrangement f^* of a function $f \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$, remains in $W^{1,p}(\mathbb{R}^n)$ and, moreover, $\|\nabla f^*\|_p \leq \|\nabla f\|_p$. In the affine Pólya-Szegő inequality the L^p norm of the gradient $\|\nabla f\|_p$ is replaced by the L^p affine energy

$$\mathcal{E}_p(f) = c_{n,p}\Big(\int_{S^{n-1}} \|D_u f\|_p^{-n} du\Big)^{-1/n},$$

where $D_u f$ is the directional derivative of f in the direction u and the constant $c_{n,p}$ is such that

$$\mathcal{E}_p(f^*) = \|\nabla f^*\|_p$$

for $f \in W^{1,p}(\mathbb{R}^n)$. Note that, unlike $\|\nabla f\|_p$, $\mathcal{E}_p(f)$ is invariant under volume preserving affine transformations on $\mathbb{R}^n$. *A. Cianchi*, *E. Lutwak*, *D. Yang* and *G. Zhang* [Calc. Var. Partial Differ. Equ. 36, No. 3, 419–436 (2009; Zbl 1202.26029)] proved the affine Pólya-Szegő principle $\mathcal{E}_p(f^*) \leq \mathcal{E}_p(f)$. *E. Lutwak*, *D. Yang* and *G. Zhang* [J. Differ. Geom. 62, No. 1, 17–38 (2002; Zbl 1073.46027)] proved that

$$\mathcal{E}_p(f) \leq \|\nabla f\|_p$$

thus showing that the affine inequality is stronger than the original Euclidean one.

The authors introduce the asymmetric L^p affine energy

$$\mathcal{E}_p^+(f) = 2^{1/p} c_{n,p}\Big(\int_{S^{n-1}} \|D_u^+ f\|_p^{-n} du\Big)^{-1/n},$$

where $D_u^+ f = \max\{D_u f, 0\}$. The asymmetric affine energy $\mathcal{E}_p^+(f)$ is again invariant under volume preserving affine transformations on $\mathbb{R}^n$, but it differs from the (symmetric) affine energy $\mathcal{E}_p(f)$ by the fact that the odd parts of the directional derivative do not vanish. The authors prove that for $p \geq 1$ and $f \in W^{1,p}(\mathbb{R}^n)$ also f^* belongs to $W^{1,p}(\mathbb{R}^n)$ and the estimate $\mathcal{E}_p^+(f^*) \leq \mathcal{E}_p^+(f)$ holds. Using this result they derive new sharp asymmetric affine versions of various inequalities like the classical Sobolev and logarithmic Sobolev inequalities, the Nash inequality, the Moser-Trudinger inequality, the Morrey-Sobolev inequality, and the Gagliardo-Nirenberg inequality.

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MSC:

26D10 Inequalities involving derivatives and differential and integral operators

46E35 Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems

Keywords:

Pólya-Szegő principle; symmetric rearrangement; asymmetric affine energy; Sobolev inequality; logarithmic Sobolev inequality; Nash inequality; Moser-Trudinger inequality; Morrey-Sobolev inequality; Gagliardo-Nirenberg inequality

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