Heumann, Holger; Hiptmair, Ralf

Convergence of lowest-order semi-Lagrangian schemes. (English) Zbl 1273.65121

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The authors consider a non-stationary advection-diffusion problem for time-dependent differential forms. By means of the Hille-Yosida theorem, the existence and the uniqueness of the transient advection-diffusion problem are obtained. The semi-Lagrangian Galerkin time-stepping scheme for the considered advection-diffusion problem is presented. Under some additional assumptions, an L^2-estimate of order $O(\tau + h^r + h^{r+1}\tau^{-1/2} + \tau^{1/2})$ is established, with h the spatial meshsize, τ the time step and r the polynomial degree of the trial functions.

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MSC:

65M12 Stability and convergence of numerical methods for initial value and initial-boundary value problems involving PDEs

65M25 Numerical aspects of the method of characteristics for initial value and initial-boundary value problems involving PDEs

65M60 Finite element, Rayleigh-Ritz and Galerkin methods for initial value and initial-boundary value problems involving PDEs

62M20 Inference from stochastic processes and prediction

35K20 Initial-boundary value problems for second-order parabolic equations

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