For a given Hilbert space operator \( A: H \to H \) an orthonormal basis \( \{ e_n \} \) of \( H \) is “well behaved with respect to \( A \)” if \( \{ Ae_n \} \) is norm bounded from below. Obviously for a compact operator \( A \) there are no well behaved orthonormal bases, and for an isomorphism \( A \), all of them are.

In this paper it is established that for a non compact, injective operator \( A \) there exist well behaved orthonormal bases \( \{ e_n \} \) such that \( \{ Ae_n \} \) is a strong-M-basis. (Actually they can even be heterogonal by blocks).

The question which sequences of one dimensional subspaces \( \{ r_n \} \) are images of well behaved orthonormal bases is answered by the equivalence of the following propositions a) and b):

a) There exists an operator \( A \) and a “well behaved the respect to \( A \)” orthonormal basis \( \{ e_n \} \) such that \( [Ae_n] = r_n \) ([ ] stands for the closed linear span) for every \( n \in N \).

b) If \( a_n \in r_n - \{ 0 \}, \forall n \in N, \sum_n \| a_n \|^2 < \infty \) if and only if \( \sum_n a_n \) is unconditionally convergent.

The sequences \( \{ r_n \} \) verifying a) or b) are called L-systems of rays (SLR).

This result is generalized to \( \ell^p \) \( (p > 2) \) by the equivalence for a sequence \( (a_n) \subset \ell^p \) with \( 0 < \inf \| a_n \| \leq \sup \| a_n \| < \infty \) of the following two conditions:

a) The sequence \( (a_n) \) is weakly p-summable.

b) \( \sum_n \lambda_n a_n \) is unconditionally convergent if and only if \( \sum |\lambda_n|^{p(p-1)} < \infty \).

For \( p < 2 \) the weakly p-summable sequences cannot be bounded from below.

MSC:

46C05 Hilbert and pre-Hilbert spaces: geometry and topology (including spaces with semidefinite inner product)
46B15 Summability and bases; functional analytic aspects of frames in Banach and Hilbert spaces
47B06 Riesz operators; eigenvalue distributions; approximation numbers, s-numbers, Kolmogorov numbers, entropy numbers, etc. of operators
40H05 Functional analytic methods in summability

Keywords:
compact operator; well behaved orthonormal bases; strong-M-basis; L-systems of rays

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