The following problem is still open: is each finite lattice isomorphic to the lattice $\mathcal{O}_G(H)$ of overgroups of some subgroup $H$ in a finite group $G$? The main theorem of this paper is a step in a program to show that the answer is negative.

For a finite lattice $\Lambda$, let $\Lambda'$ be the poset obtained by removing the minimum and maximum elements of $\Lambda$ and, for $n \in \mathbb{N}$, let $\Delta(n)$ be the lattice of all subsets of $\{1, \ldots, n\}$, ordered by inclusion. A lattice $\Lambda$ is a $D\Delta$-lattice if $\Lambda$ is disconnected and $\Lambda'$ has connected components $\Lambda_i$, $1 \leq i \leq k$, such that $\Lambda_i \cong \Delta(m_i)'$ for some $m_i \geq 3$.

Shareshian and the author conjectured that $D\Delta$-lattices are not overgroup lattices in any finite group. In two previous papers the author obtained a reduction of this conjecture to two problems about almost simple groups. The first is to show that if $G$ is an almost simple finite group and $H \leq G$ then $\mathcal{O}_G(H)$ is not a $D\Delta$-lattice. The second is to show that a lower signalizer lattice in an almost simple finite group is not a $D\Delta$-lattice. Both problems have been solved when $G$ is an alternating or symmetric group.

The present paper is a first step to deal with the first question when the socle of $G$ is a group of Lie type: it seems reasonable to approach this question by successively eliminating the possibility that the set of maximal overgroups of $H$ in $G$ contains a member of each of the various collections of maximal subgroups of $G$. The largest subgroups of $G$ are its parabolics. The main theorem shows $H$ cannot have an overgroup in that collection. As a part of the proof, the author proves a theorem on the structure of maximal parabolics in finite groups of Lie type which is of independent interest.

Reviewer: Andrea Lucchini (Padova)

MSC:

20D30 Series and lattices of subgroups
20D06 Simple groups: alternating groups and groups of Lie type
06B15 Representation theory of lattices
20G40 Linear algebraic groups over finite fields
08A30 Subalgebras, congruence relations

Keywords:
finite groups; finite groups of Lie type; subgroup lattices; intervals; parabolic subgroups; almost simple groups; signalizer lattices

Full Text: DOI

References:
