Nyman, A.

The geometry of arithmetic noncommutative projective lines. (English) [Zbl 1307.14002]


Let $k$ be a perfect field, and let $K/k$ be a finite extension. This article studies the geometry of noncommutative spaces (Grothendieck categories) of the form $\text{Proj} S_K(V)$ where $V$ is a $k$-central $K - K$-bimodule which is two-dimensional as a vector space via the right and left actions of $K$ on $V$. $\text{Proj} S_K(V) = P_K(V)$ denotes the quotient of the category of graded right $S_K(V)$-modules modulo the full subcategory of direct limits of right bounded modules. The space $P_K(V)$ is called the arithmetic projective line, and one of the main issues of this article is to show that its geometry has close connection to data associated with $K/k$. The noncommutative projective lines serves as basic examples in the field of noncommutative arithmetic geometry.

Artin’s conjecture states that the division ring of fractions of a noncommutative surface, not finite over its center, is the function field of a noncommutative projective line. It is proved that these spaces are noetherian, Ext-finite, homological dimension one categories, also having a Serre functor. The first main result states that noncommutative projective lines are integral (in a defined sense), that the line bundles over a noncommutative projective line are indexed by $\mathbb{Z}$, and that every vector bundle over a noncommutative projective line is a direct sum of line bundles.

Using the classification of vector bundles, the author classifies noncommutative projective lines up to $k$-linear equivalence. The statement is that there is a $k$-linear equivalence $P_K(V) \to P_K(W)$ if and only if there exists $a \in \text{Gal}(K/k)$ such that $\delta V \cong K \otimes K \sigma_1 \otimes K \sigma_2$. In cases $V \cong K \otimes K \sigma_1 \otimes K \sigma_2$, and under the action of $\text{Gal}(K/k)$ on itself defined by $(\sigma, \tau)(\delta, \epsilon) = (\delta^{-1} \sigma, \delta^{-1} \tau \epsilon)$, contains an element of the form $\{(\sigma_2, \tau_2), (\sigma_2^{-1}, \tau_2^{-1}), (\tau_2, \sigma_2), (\tau_2^{-1}, \sigma_2^{-1})\}$.

It is known that there is a one-to-one correspondence between the finite orbits $\text{Emb}(K)$ under the action of $G = \text{Gal}(K/k)$ and the isomorphism classes of simple left finite dimensional two-sided vector spaces. This allows the author to sharpen the classification of noncommutative projective lines: For char $k \neq 2$ and $V_i$ a rank 2 two-sided vector space for $i = 1, 2$, there is a $k$-linear equivalence $P_K(V_1) \to P_K(V_2)$ if and only if (1) there exists $\sigma_i \in \text{Gal}(K/k)$ such that $V_i \cong K \otimes K \sigma_i$. In this case, $P(V_i)$ is equivalent to the commutative projective line over $K$, (2) there exists $\sigma_1, \sigma_2 \in \text{Gal}(K/k)$ with $\sigma_1 \neq \sigma_2$, and $V_i \cong K \otimes K \sigma_i$, and under the action of $\text{Gal}(K/k)^2$ on itself defined by $(\sigma, \tau)(\delta, \epsilon) = (\delta^{-1} \sigma, \delta^{-1} \tau \epsilon)$, the orbit of $(\sigma_1, \tau_1)$ contains an element of the form $\{(\sigma_2, \tau_2), (\sigma_2^{-1}, \tau_2^{-1}), (\tau_2, \sigma_2), (\tau_2^{-1}, \sigma_2^{-1})\}$, (3) $V_i \cong V(\lambda_i)$, and under the action of $\text{Gal}(K/k)^2$ on $\Lambda(K)$ defined by $\lambda^G \cdot (\delta, \epsilon) = (\delta^{-1} \lambda \epsilon)^G$, the orbit of $\lambda^G$ contains either $\lambda^G$ or $\mu^G$. $\delta$ denotes an extension of $\delta$ to $K$.

The author gives the definition of the three types of canonical equivalences between noncommutative projective lines. Then the classification of $k$-linear equivalences $P_K(V) \to P_K(W)$ up to isomorphism can be given for char $k \neq 2$: Let $F : P_K(V) \to P_K(W)$ be a $k$-linear equivalence. Then there exists $\delta : \text{Gal}(K/k)$, an isomorphism $\phi : K \otimes K V \otimes K K \to W^\ast$, an integer $i$ such that $F[-i] \circ \phi \circ \delta^i$. Also, $\delta, \epsilon, i$ are unique up to natural equivalence, while $\Phi$ is naturally equivalent to $\Phi'$ if and only if there exist nonzero $a, b, c \in K$ such that $a w = a \cdot w \cdot b$ for all $w \in W^\ast$.

The last result is a computation of the automorphism group of a noncommutative projective line. The automorphism group of $P_K(V)$ is the set of $k$-linear shift-free equivalences $P_K(V) \to P_K(W)$ modulo natural equivalence.
Also, the author discusses some of the other notions of noncommutative projective line given in the literature. For some of these notions, the results in this article proves that a noncommutative projective line is a commutative curve of genus 0, and necessary and sufficient conditions for noncommutative curves of genus zero to be noncommutative projective lines are given. The theory presented in this article, just don’t fit into other notions.

All necessary definitions and explicit computations are given in the text. The article is a long and thorough explanation of the geometry of noncommutative projective lines, and the arithmetic of these lines are explicit.

Reviewer: Arvid Siqveland (Kongsberg)

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References:


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