Griffiths, Phillip A.; **Jensen, Gary R.**

Differential systems and isometric embeddings. The William H. Roever lectures in geometry, Washington University, St. Louis. (English) Zbl 0637.53001

Annals of Mathematics Studies, No. 114. Princeton, New Jersey: Princeton University Press. XII, 225 p.; Cloth: $ 35.00; Paper: $ 15.00 (1987).

An isometric embedding of a smooth n-dimensional Riemannian manifold M into an N-dimensional Euclidean space $\mathbb{R}^N$is a smooth injective map $F : M \to \mathbb{R}^N$which preserves inner products; i.e., the components F^α of F satisfy

$$(*) \quad \sum_{\alpha=1}^{N} \frac{\partial F^\alpha}{\partial y \quad i} \frac{\partial F^\alpha}{\partial y \quad j} = g \quad _{ij},$$

where $g_{ij}(y)$ are components of the Riemannian metric ds 2 in local coordinates. Much is known about the local isometric embedding problem (i.e., the existence and uniqueness problem for the nonlinear system (*)); see, for example, *R. E. Greene* [Mem. Am. Math. Soc. 97, 63 p. (1970; Zbl 0203.240)], *M. L. Gromov* and *V. A. Rokhlin* [Usp. Mat. Nauk 25, No.5(155), 3-62 (1970; Zbl 0202.210); English translation Russ. Math. Surv. 25, No.5, 1-57 (1970)] or *M. Spivak* [A comprehensive introduction to differential geometry, Vol. 5 (especially chapters 11 and 12), Publish or Perish, Inc. VII (1979; Zbl 0439.53005)]. For instance, local analytic solutions exist if (M,ds 2) is real analytic and N equals the 'critical dimension' $E(n) = n(n+1)/2$ (Cartan-Janet- Burstin-Schlaefli); in the C^∞ case, smooth local solutions exist provided $N \geq N(n)+n$ (Nash-Greene). As for uniqueness, a 'generic' local isometric embedding is rigid (i.e., unique up congruences in $\mathbb{R}^N$) provided N-n$\leq [n/3]$ (Allendoerfer-Beez).

The book under review gives an exposition of recent developments, particularly those connected with the characteristic variety, in the theory of Pfaffian differential systems (PDSs), focussing on the applications to local isometric embeddings. (A PDS is a pair of sub- bundles I and J of the cotangent bundle T *M such that $I \subset J$; this is dual to a pair of distributions $\Delta_I = I^\perp$ and $\Delta_J = J^\perp$ with $\Delta_J \subset \Delta_I$.) After reviewing the literature on isometric embeddings and the structure equations for submanifolds of Euclidean space (Chapters 1 and 2), the authors give, in Chapter 3, a quick course on Pfaffian differential systems including involution, prolongation, the Cartan-Kähler theorem and, in an appendix, the Cauchy characteristic space.

Chapter 4 specializes to the case of 'quasi-linear' PDSs (i.e., pairs (I,J) such that d$\mathcal{I} \subset \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ are the algebraic ideals in $C^\infty(T$ *M) generated by I and J, respectively). This is not a great restriction, since prolonged systems are automatically 'quasi-linear'. In Chapter 5 the authors apply the preceding theory to the PDS which arises naturally from the isometric embedding problem. Then in Chapter 6, after developing the theory of the characteristic variety, they devote nearly thirty pages to a sketch of the proof of their main result, the rigidity theorem of *E. Berger*, *R. Bryant* and *P. Griffiths* [Duke Math. J. 50, 803-892 (1983; Zbl 0526.53018)]. Finally, Chapters 7 and 8 involve special applications: isometric embedding of space forms, and embedding Cauchy-Riemann structures.

This book is an elaboration of a series of lectures which the senior author gave at Washington University in January, 1984, and it maintains the feel of 'spoken' mathematics; in particular, the authors frequently omit technical details in definitions and proofs, preferring to illustrate the main ideas with prototypical examples and then to refer readers wishing more precision to the literature. (The crucial reference, which they repeatedly cite, is the book 'Essays on exterior differential systems' by Bryant, Chern, Gardner, Goldschmidt, Griffiths, and Yang; it is listed as being 'in preparation'.) The writing is generally clear and the errors seem to be few and unimportant. (For instance: the bibliography is missing the reference to *S.-S. Chern* and *R. Osserman* [Geometry, Proc. Symp., Utrecht 1980, Lect. Notes Math. 894, 49-90 (1981; Zbl 0477.53056)] cited on page 9; the symbol PV near the bottom of page 112 should be PV *)

{Reviewer's Remark: The 'quasi-precise' approach which the authors follow is perfectly suited to a short series of lectures. In a monograph as long as the one under review, however, the lack of precision gradually begins to weigh on the reader, especially when the main reference for the details is not yet in print.}

Reviewer: R.-C.Reilly

MSC:

53-02 Research exposition (monographs, survey articles) pertaining to differential geometry

Cited in **8** Documents

53B25 Local submanifolds

53C40 Global submanifolds

58A17 Pfaffian systems

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Full Text: DOI