Goresky, Mark; MacPherson, Robert
Stratified Morse theory. (English) Zbl 0639.14012
Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 14. Berlin etc.: Springer-Verlag. XIV, 272 p.; DM 148.00 (1988).

This important and widely expected monograph deals with some far reaching extensions of Morse theory. Classically, this theory relates the topology of a smooth manifold M to the indices of the singularities of a Morse function $f : M \to R$ (i.e. f is a smooth function with distinct critical values, all the singularities of f are nondegenerate and the level sets $M_{\leq a} = f^{-1}(-\infty, a]$ are compact). More precisely, if the set $f^{-1}[a, b]$ contains exactly one singular point, say $x \in M$ with index λ, then $M_{\leq b}$ is homeomorphic to the space obtained from $M_{\leq a}$ by attaching a λ-handle $(A, B) = (D^{n-\lambda} \times D^\lambda, D^{n-\lambda} \times \partial D^\lambda)$, where $n = \dim(M)$.

In order to study the topoloy of quite usual spaces (e.g. complex algebraic sets) it is necessary to replace the manifold M by a Whitney stratified space and to find the corresponding extension for the notion of Morse function. - After doing this, the first basic result of the book says that the space $M_{\leq b}$ is still homeomorphic to a space obtained from $M_{\leq a}$ by glueing in a certain space A along a closed subspace B and that the pair (A,B), called local Morse data at x, depends only on the function germ $f : (M, x) \to (R, f(x))$. This pair (A,B) has in general a complicated structure and the second main result gives a splitting up to homeomorphism:

$$\text{local Morse data} = \text{normal Morse data} \times \text{tangential Morse data},$$

where tangential (resp. normal) Morse data at the point x is the local Morse data at x of the restriction of f to the stratum in M containing the point x (resp. to the normal slice to this stratum in M).

This gives an inductive way to determine the local Morse data and finally the topology of M. All this and much more is contained in the first part of this book ($\simeq$ 140 pages).

The second part ($\simeq$ 90 pages) makes a careful analysis of the case when M is a complex analytic space with an analytic stratification. Much more can be said in this case about the tangential and normal Morse data (e.g. the latter can be related in a subtle way to other basic objects such as Milnor fibers, complex links and polar varieties). - Using their extension of Morse theory, the authors get very general and powerful local and global Lefschetz theorems as well as bounds on the homotopy dimension of some complex spaces (extending and giving a new proof for the known fact that an n-dimensional Stein space has homotopy dimension at most n). - The topological consequences are stated in terms of homotopy, homology and intersection homology groups, the latter ones being sometimes obviously the most natural framework (e.g. the classical notion of index of a singular point has an analog in this general setting only when using intersection homology!).

The last part ($\simeq$ 20 pages) investigates the topology of complements of unions of linear subspaces in affine or projective (real and complex) spaces. In spite of the fact that these complements are smooth, their study is based on a further extension of Morse theory due to the authors, the so called relative Morse theory of nonproper functions.

The proofs of the main results in the book are long and difficult but the arguments are made more transparent by the use of the authors' innovations (intuitive and rigorous in the same time notions) of moving the wall and fringed set.

In conclusion, the book is an important step forward in understanding the topology of singular o

MSC:

14F45	Topological properties in algebraic geometry
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry
57-02	Research exposition (monographs, survey articles) pertaining to manifolds and cell complexes
14F35	Homotopy theory and fundamental groups in algebraic geometry
57R70	Critical points and critical submanifolds in differential topology
58-02	Research exposition (monographs, survey articles) pertaining to global analysis
58E05	Abstract critical point theory (Morse theory, Lyusternik-Shnirel'man theory, etc.) in infinite-dimensional spaces
14F99	(Co)homology theory in algebraic geometry

Keywords:

topology of singular spaces; local Morse data; Lefschetz theorems; bounds on the homotopy dimension; intersection homology; topology of complements of unions of linear subspaces; relative Morse theory