Dartyge, Cécile

Chebyshev's problem for the twelfth cyclotomic polynomial. (Le problème de Tchébychev pour le douzième polynôme cyclotomique.) (French. English summary) Zbl 1323.11069

Proc. Lond. Math. Soc. (3) 111, No. 1, 1-62 (2015).

It was shown by Chebyshev that there are infinitely many n for which the largest prime factor of n^2+1 is strictly larger than n. This was strengthened by *C. Hooley* [Acta Math. 117, 281–299 (1967; Zbl 0146.05704)], who showed that the largest prime factor is at least $n^{11/10}$ infinitely often. The reviewer [Proc. Lond. Math. Soc. (3) 82, No. 3, 554–596 (2001; Zbl 1023.11048)] proved a corresponding result for the largest prime factor p of n^3+2, showing that there is a constant $\delta > 0$ such that $p > n^{1+\delta}$ for infinitely many n. Indeed one may take $\delta = 10^{-303}$. The papers by Hooley and the reviewer use incomplete exponential sums. In Hooley's case the length of the exponential sum is comparable with its modulus, but for n^3+2 the length is around the square-root of the modulus. However, one is able to arrange that the modulus factors sufficiently well for the q-analogue of van der Corput's method to be used.

The present paper handles a quartic polynomial for the first time. It is shown that there are infinitely many n for which the largest prime factor of n^4-n^2+1 is at least $n^{1+\delta}$, where one can take $\delta = 10^{-26531}$. The proof follows the same basic line of attack as in the reviewer's work. However there is a major obstacle at the very start, when one expresses the problem in terms of exponential sums. The reviewer's procedure is somewhat *ad hoc* and it is by no means clear how to extend it to higher degree polynomials, or indeed whether such a generalization exists. Thus it is the corresponding argument in the present paper which is the most important forward step. The author makes heavy use of the explicit shape of the polynomial X^4-X^2+1, and it is not at all clear to what extent it will generalize to other quartic polynomials.

In addition to the exponential sums which occur it is necessary to handle sieving conditions involving lattices in $\mathbb{Z}^3$. The paper provides a level-of-distribution result in this context, somewhat in the spirit of *S. Daniel* [J. Reine Angew. Math. 507, 107–129 (1999; Zbl 0913.11041)] and *G. Marasingha* [J. Lond. Math. Soc., II. Ser. 82, No. 2, 295–316 (2010; Zbl 1269.11099)]. The sieve argument can then be completed along the lines of the reviewer's paper.

Reviewer: D. R. Heath-Brown (Oxford)

MSC:

11N32	Primes represented by polynomials; other multiplicative structures of polynomial values
11N36	Applications of sieve methods
11P21	Lattice points in specified regions
11L07	Estimates on exponential sums
11L26	Sums over arbitrary intervals

Keywords:

prime; polynomial; quartic; largest prime factor; cyclotomic polynomial

Full Text: DOI

References:

[1] Balog A. Blomer V. Dartyge C. Tenenbaum G. , 'Friable values of binary forms', Comment. Math. Helv. 87 (2012) 639–667. · Zbl 1268.11136 · doi:10.4171/CMH/264

[2] DOI: 10.5802/aif.1535 · Zbl 0853.11088 · doi:10.5802/aif.1535

[3] DOI: 10.1215/S0012-7094-99-09807-1 · Zbl 1061.11503 · doi:10.1215/S0012-7094-99-09807-1

[4] Benedetti R. Risler J.-J. , Real algebraic and semi-algebraic sets (Hermann, Paris, 1990). · Zbl 0694.14006

[5] DOI: 10.4064/aa125-3-6 · Zbl 1159.11035 · doi:10.4064/aa125-3-6

[6] DOI: 10.1112/S0010437X08003692 · Zbl 1234.11132 · doi:10.1112/S0010437X08003692

[7] de la Bretèche R. Browning T. D. , 'Le problème des diviseurs pour des formes binaires de degré 4', J. reine angew. Math. 646 (2010) 1–44. · Zbl 1204.11158 · doi:10.1515/crelle.2010.064

[8] DOI: 10.1007/s11856-012-0019-y · Zbl 1293.11058 · doi:10.1007/s11856-012-0019-y

[9] DOI: 10.1112/S0025579311002154 · Zbl 1284.11126 · doi:10.1112/S0025579311002154

[10] de la Bretèche R. Tenenbaum G. , 'Sur la conjecture de Manin pour certaines surfaces de Châtelet', J. Inst. Math. Jussieu 12 (2013) 719–759. · Zbl 1295.11029 · doi:10.1017/S1474748012000886

[11] Cassels J. W. S. , An introduction to the geometry of numbers, Die Grundlehren der Mathematischen Wissenschaften Band 99 (Springer, Berlin, 1959). · Zbl 0086.26203 · doi:10.1007/978-3-642-62035-5

[12] Daniel S. , 'On the divisor-sum problem for binary forms', J. reine angew. Math. 507 (1999) 107–129. · Zbl 0913.11041 · doi:10.1515/crll.1999.507.107

[13] Dartyge C. , 'Le plus grand facteur premier de \(n^2+1\) où \(n\) est presque premier', Acta Arith. 76 (1996) 199–226.

[14] Davenport H. , 'On a principle of Lipschitz', J. London. Math. Soc. 26 (1951) 179–183. · Zbl 0042.27504 · doi:10.1112/jlms/s1-26.3.179

[15] DOI: 10.1007/BF01390728 · Zbl 0502.10021 · doi:10.1007/BF01390728

[16] Deshouillers J.-M. Iwaniec H. , 'On the greatest prime factor of \(n^2+1\) ', Ann. Inst. Fourier (\(Grenoble\)) 32 (1982/83) 1–11. · Zbl 0489.10038

[17] Erdos P. , 'On the greatest prime factor of \(\prod _{k=1}^xf(k)\) ', J. London Math. Soc. 27 (1952) 379–384.

[18] Erdos P. Schinzel A. , 'On the greatest prime factor of \(\prod _{k=1}^xf(k)\) ', Acta Arith. 55 (1990) 191–200.

[19] Fouvry E. Iwaniec H. , 'Gaussian primes', Acta Arith. 79 (1997) 249–287. · Zbl 0881.11070

[20] DOI: 10.2307/121034 · Zbl 0926.11068 · doi:10.2307/121034

[21] Gomez C. Salvy B. Zimmermann P. , Calcul formel: mode d'emploi, exemples en Maple, 2nd edn, Logique mathématiques informatique (Masson, Paris, 1995).

[22] DOI: 10.1016/0022-314X(71)90049-7 · Zbl 0214.30301 · doi:10.1016/0022-314X(71)90049-7

[23] Halberstam H. Richert H. , Sieve methods, 2nd edn (Dover, London, 2011). · Zbl 0298.10026

[24] DOI: 10.1112/plms/82.3.554 · Zbl 1023.11048 · doi:10.1112/plms/82.3.554

[25] DOI: 10.1007/BF02392715 · Zbl 1007.11055 · doi:10.1007/BF02392715

[26] Heath-Brown D. R. , 'Linear relations amongst sums of two squares', Number theory and algebraic geometry, London Mathematical Society Lecture Note Series 303 (Cambridge University Press, Cambridge, 2003) 133–176. · Zbl 1161.11387

[27] DOI: 10.1112/plms/84.2.257 · Zbl 1030.11046 · doi:10.1112/plms/84.2.257

[28] DOI: 10.1112/S0024611503014497 · Zbl 1099.11050 · doi:10.1112/S0024611503014497

[29] DOI: 10.1017/S0305004111000752 · Zbl 1255.11048 · doi:10.1017/S0305004111000752

[30] DOI: 10.1007/BF02395047 · Zbl 0146.05704 · doi:10.1007/BF02395047

[31] Hooley C. , Application of sieve methods to the theory of numbers (Cambridge University Press, Cambridge, 1976) xiv, 122 p. · Zbl 0327.10044

[32] Hooley C. , 'On the greatest prime factor of a cubic polynomial', J. reine angew. Math. 303/304 (1978) 21–50. · Zbl 0391.10028

[33] Iwaniec H. , 'Primes represented by quadratic polynomials in two variables', Acta Arith. 24 (1974) 435–459. · Zbl 0271.10043

[34] DOI: 10.1007/BF01578070 · Zbl 0389.10031 · doi:10.1007/BF01578070

[35] Iwaniec H. , 'Rosser's sieve', Acta Arith. 36 (1980) 171–202. · Zbl 0435.10029

[36] Iwaniec H. , 'A new form of the error term in the linear sieve', Acta Arith. 37 (1980) 307–320. · Zbl 0444.10038

[37] Lang S. , Algebraic number theory, 2nd edn, Graduate Texts in Mathematics 110 (Springer, Berlin, 1994). · Zbl 0811.11001 · doi:10.1007/978-1-4612-0853-2

[38] LejeuneDirichlet G. , Mathematische Werke. Bände I, II, Herausgegeben auf Veranlassung der Königlich Preussischen Akademie der Wissenschaften von L. Kronecker (Chelsea Publ. Co., Bronx, NY, 1969).

[39] DOI: 10.4064/aa151-3-2 · Zbl 1256.11049 · doi:10.4064/aa151-3-2

[40] Marasingha G. , 'On the representation of almost primes by pairs of quadratic forms', Acta Arith. 124.4 (2006) 327–355. · Zbl 1146.11047 · doi:10.4064/aa124-4-3

[41] DOI: 10.1112/jlms/jdq026 · Zbl 1269.11099 · doi:10.1112/jlms/jdq026

[42] Markov A. A. , 'Über die Primteiler des Zahlen von der Form \(1+4x^2\) ', Bull. Acad. Sci. St. Petersburg 3 (1895) 55–59.

[43] Martinet J. , Les réseaux parfaits des espaces euclidiens, 2nd edn (Masson, Paris, 1996). · Zbl 0869.11056

[44] Nagell T. , 'Généralisation d'un théorème, Tchebycheff', J. Math. 4 (1921) 343–356.

[45] Nagell T. , Introduction to number theory (Wiley, New York, 1951). · Zbl 0042.26702

[46] DOI: 10.1007/BF02392880 · Zbl 0917.11048 · doi:10.1007/BF02392880

[47] Pólya G. , Généralisation d'un théorème de M. Störmer, Archiv for Mathematik og Naturvidenskab, t XXXV (Kristiania, 1917).

[48] Schinzel A. Sierpiński W. , 'Sur certaines hypothèses concernant les nombres premiers', Acta Arith. 4 (1958) 185–208. · Zbl

0082.25802

[49] Smith H. J. S. , 'Report on the theory of numbers', Collected Mathematical Papers, vol. 1, reprinted (Chelsea Pub. Co., New York, 1965).

[50] DOI: 10.1007/BF01234418 · Zbl 0699.10063 · doi:10.1007/BF01234418

[51] Tenenbaum G. , Introduction à la théorie analytique et probabiliste des nombres, 3rd edn (Collection Échelles, Édition Belin, 2008).

[52] Washington L. C. , 'Introduction to cyclotomic fields', Graduate Texts in Mathematics 83 (Springer, Berlin, 1982). · Zbl 0484.12001 · doi:10.1007/978-1-4684-0133-2

[53] Weil A. , 'Sur les courbes algébriques et les variétés qui s'en déduisent', Publ. Inst. Math. Univ. Strasbourg 7 (Hermann, Paris, 1948).