Fawzi, Hamza; Gouveia, João; Parrilo, Pablo A.; Robinson, Richard Z.; Thomas, Rekha R.
Positive semidefinite rank. (English) Zbl 1327.90174
Math. Program. 153, No. 1 (B), 133-177 (2015).

Summary: Let $M \in \mathbb{R}^{p\times q}$ be a nonnegative matrix. The positive semidefinite rank (psd rank) of M is the smallest integer k for which there exist positive semidefinite matrices A_i, B_j of size $k\times k$ such that $M_{ij} = \operatorname{trace}(A_iB_j)$. The psd rank has many appealing geometric interpretations, including semidefinite representations of polyhedra and information-theoretic applications. In this paper we develop and survey the main mathematical properties of psd rank, including its geometry, relationships with other rank notions, and computational and algorithmic aspects.

MSC:

90C22 Semidefinite programming
15A23 Factorization of matrices
68Q17 Computational difficulty of problems (lower bounds, completeness, difficulty of approximation, etc.)

Full Text: DOI arXiv

References:

[1] Arora, S., Ge, R., Kannan, R., Moitra, A.: Computing a nonnegative matrix factorization-provably. In: Proceedings of the 44th Symposium on Theory of Computing (STOC), ACM, pp. 145-162 (2012) · Zbl 1286.15014

[2] Barvinok, A, A remark on the rank of positive semidefinite matrices subject to affine constraints, Discret. Comput. Geom., 25, 23-31, (2001) · Zbl 0969.90096

[3] Barvinok, A.: Approximations of Convex Bodies by Polytopes and by Projections of Spectrahedra. arXiv preprint. arXiv:1204.0471 (2012) · Zbl 1202.94012

[4] Berman, A., Shaked-Monderer, N.: Completely Positive Matrices. World Scientific Pub Co Inc, Singapore (2003) · Zbl 1030.15022

[5] Blekherman, G., Parrilo, P.A., Thomas, R. (eds.): Semidefinite Optimization and Convex Algebraic Geometry, vol 13 of MOS-SIAM Series on Optimization. SIAM (2012) · Zbl 1260.90006

[6] Bocci, C; Carlini, E; Rapallo, F, Perturbation of matrices and nonnegative rank with a view toward statistical models, SIAM J. Matrix Anal. Appl., 32, 1500-1512, (2011) · Zbl 1242.15031

[7] Boyd, S.P., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge (2004) · Zbl 1058.90049

[8] Briët, J., Dadush, D., Pokutta, S.: On the existence of 0/1 polytopes with high semidefinite extension complexity. Algorithms. ESA 2013. Lecture Notes in Computer Science. vol 8125, pp. 217-228. Springer, Berlin (2013) · Zbl 1395.90241

[9] Brunner, N; Pironio, S; Acin, A; Gisin, N; Méthot, AA; Scarani, V, Testing the dimension of Hilbert spaces, Phys. Rev. Lett., 100, 210503, (2008) · Zbl 1228.81056

[10] Burgdorf, S., Laurent, M., Piovesan, T.: On the closure of the completely positive semidefinite cone and linear approximations to quantum colorings. arXiv preprint arXiv:1502.02842 (2015) · Zbl 1372.81026

[11] Cohen, JE; Rothblum, UG, Nonnegative ranks, decompositions, and factorizations of nonnegative matrices, Linear Algebra Appl., 190, 149-168, (1993) · Zbl 0784.15001

[12] Dur, M.: Copositive programming-a survey. In: Diehl., M. G., Francois, J., Elias, M. W. (eds.) Recent advances in optimization and its applications in engineering, Springer, Berlin, Heidelberg, pp. 3-20 (2010). doi:10.1007/978-3-642-12598-0_1 · Zbl 0845.65023

[13] Fawzi, H., Saunderson, J., Parrilo, P.A.: Sparse Sum-of-Squares Certificates on Finite Abelian Groups. arXiv preprint. arXiv:1503.01207 (2015) · Zbl 1327.90175

[14] Fiorini, S., Massar, S., Pokutta, S., Tiwary, H.R., de Wolf, R.: Linear vs. semidefinite extended formulations: exponential separation and strong lower bounds. In: Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing, STOC '12, pp 95-106. ACM (2012) · Zbl 1286.90125

[15] Fiorini, S; Kaibel, V; Pashkovich, K; Theis, DO, Combinatorial bounds on nonnegative rank and extended formulations, Discrete Math., 313, 67-83, (2013) · Zbl 1259.90111

[16] Frenkel, PE; Weiner, M, On vector configurations that can berealized in the cone of positive matrices, Linear Alg. Appl., 459, 465-474, (2014) · Zbl 1309.15049

[17] Garey, M.R., Johnson, D.S.: Computers and Intractability: A guide to the Theory of NP-Completeness. W. H Freeman and Company, San Francisco (1979) · Zbl 0411.68039

[18] Gillis, N; Glineur, F, On the geometric interpretation of the nonnegative rank, Linear Algebra Appl., 437, 2685-2712, (2012) · Zbl 1258.65039

[19] Goemans, M. X.: Smallest compact formulation for the permutahedron. Math. Prog. (2014). pp. 1-7 doi:10.1007/s10107-014-0757-1 · Zbl 1202.94012

[20] Gouveia, J; Parrilo, PA; Thomas, RR, Lifts of convex sets and cone factorizations, Math. Operat. Res., 38, 248-264, (2013) · Zbl 1291.90172

[21] Gouveia, J; Robinson, RZ; Thomas, RR, Polytopes of minimum positive semidefinite rank, Discrete Comput. Geom., 50, 679-699, (2013) · Zbl 1279.52023

[22] Gouveia, J., Fawzi, H., Robinson, R.Z.: Rational and Real Positive Semidefinite Rank can be Different. arXiv preprint. arXiv:1404.4864 (2014)

[23] Gouveia, J., Robinson, R.Z., Thomas, R.R.:Worst-case results for positive semidefinite rank. Math. Program. 1-12 (2015). doi:10.1007/s10107-015-0867-4 · Zbl 1344.90046

[24] Hrubeš, P, On the nonnegative rank of distance matrices, Inf. Process. Lett., 112, 457-461, (2012) · Zbl 1243.15003

[25] Jain, R; Shi, Y; Wei, Z; Zhang, S, Efficient protocols for generating bipartite classical distributions and quantum states, IEEE Trans. Inf. Theory, 59, 5171-5178, (2013) · Zbl 1364.81061

[26] Jolliffe, I.: Principal Component Analysis, 2nd edn. Springer, New York (2002) · Zbl 1011.62064

[27] Kalman, RE, Mathematical description of linear dynamical systems. J. soc. ind. appl. math. ser. A, Control, 1, 152-192, (1963) · Zbl 0145.34301

[28] Laurent, M., Piovesan, T.: Conic Approach to Quantum Graph Parameters Using Linear Optimization Over the Completely Positive Semidefinite Cone. arXiv preprint. arXiv:1312.6643 (2013)

[29] Lee, DD; Seung, HS, Learning the parts of objects by non-negative matrix factorization, Nature, 401, 788-791, (1999) · Zbl 1369.68285

[30] Lee, T., Theis, D.O.: Support-Based Lower Bounds for the Positive Semidefinite Rank of a Nonnegative Matrix. arXiv preprint. arXiv:1203.3961 (2012)

[31] Lee, J. R., Raghavendra, P., Steurer, D.: Lower Bounds on the Size of Semidefinite Programming Relaxations. arXiv preprint. arXiv:1411.6317 (2014) · Zbl 1321.90099

[32] Lee, T., Wei, Z., de Wolf, R.: Some Upper and Lower Bounds on Psd-rank arXiv preprint. arXiv:1407.4308 (2014)

[33] Linial, N; Shraibman, A, Lower bounds in communication complexity based on factorization norms, Random Struct. Algorithms, 34, 368-394, (2009) · Zbl 1202.94012

[34] Linial, N; Mendelson, S; Schechtman, G; Shraibman, A, Complexity measures of sign matrices, Combinatorica, 27, 439-463, (2007) · Zbl 1164.68006

[35] Lim, L-H; Comon, P, No article title, Nonnegative approximations of nonnegative tensors. J. Chemometrics, 23, 432-441, (2009)

[36] Moitra, A.: An Almost Optimal Algorithm for Computing Nonnegative Rank. In: Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposiumon Discrete Algorithms. chap. 104, pp. 1454-1464 (2013). doi:10.1137/1.9781611973105.104

[37] Mond, D; Smith, J; Straten, D, Stochastic factorizations, sandwiched simplices and the topology of the space of explanations, R. Soc. Proc. Math. Phys. Eng. Sci., 459, 2821-2845, (2003) · Zbl 1051.60076

[38] Moore, B, Principal component analysis in linear systems: controllability, observability, and model reduction, IEEE Trans. Autom. Control, 26, 17-32, (1981) · Zbl 0464.93022

[39] Nie, J; Ranestad, K; Sturmfels, B, The algebraic degree of semidefinite programming, Math. Program., 122, 379-405, (2010) · Zbl 1184.90119

[40] Pashkovich, K.: Extended formulations for combinatorial polytopes. PhD thesis, Otto-von-Guericke-Universität Magdeburg (2012) · Zbl 0969.90096

[41] Pataki, G, On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues, Math. Operat. Res., 23, 339-358, (1998) · Zbl 0977.90051

[42] Pólik, I; Terlaky, T, A survey of the S-lemma, SIAM review., 49, 371-418, (2007) · Zbl 1128.90046

[43] Renegar, J, On the computational complexity and geometry of the first-order theory of the reals. part I: introduction. preliminaries. the geometry of semi-algebraic sets. the decision problem for the existential theory of the reals, J. Symb. Comput., 13, 255-299, (1992) · Zbl 0763.68042

[44] Renegar, J, Hyperbolic programs, and their derivative relaxations, Found. Comput. Math., 6, 59-79, (2006) · Zbl 1130.90363

[45] Scheiderer, C.: Semidefinite representation for convex hulls of real algebraic curves. arXiv preprint. arXiv:1208.3865 (2012) · Zbl 1364.81061

[46] Stark, C. J., Harrow, A. W.: Compressibility of positive semidefinite factorizations and quantum models. arXiv preprint arXiv:1412.7437 (2014) · Zbl 1359.94175

[47] Vandenberghe, L; Boyd, S, Semidefinite programming, SIAM Rev., 38, 49-95, (1996) · Zbl 0845.65023

[48] Vavasis, SA, On the complexity of nonnegative matrix factorization, SIAM J. Optim., 20, 1364-1377, (2009) · Zbl 1206.65130

[49] Wehner, S; Christandl, M; Doherty, AC, Lower bound on the dimension of a quantum system given measured data, Phys.

Rev. A, 78, 062112, (2008)

[50] Yannakakis, M, Expressing combinatorial optimization problems by linear programs, J. Comput., 43, 441-446, (1991) · Zbl 0748.90074