Thaine, Francisco

On the ideal class groups of real abelian number fields. (English) Zbl 0665.12003

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This paper presents a new relationship between the ideal class group A and the unit group E of a real abelian field K. In fact, the author finds annihilators of the p-class group $(A)_p$ related to the structure of $W = E/C$ (C = circular units), and in this way obtains smooth statements about relations between annihilators of $(A)_p$ and $(W)_p$.

Let m denote the conductor of K, let ζ_n be a primitive nth root of unity, and put $\Delta = \mathrm{Gal}(K/\mathbb{Q})$. Define the group C of circular units of K to consist of all units of the form $f(1)$, where

$$f(X) = \pm \prod_{i=1}^{j} \prod_{k=1}^{m-1} (X^i - \zeta_m^k)^{a_{ik}} \in K(X)$$

with $a_{ik} \in \mathbb{Z}$ and $j \geq 1$ (the author points out that this definition differs from the known and may produce a larger group). Consider first a field $L = K(\zeta_q)$, where q is a sufficiently large prime that splits completely in K. Starting with a unit $\delta = f(1) \in C$ the author observes that a generating automorphism of L/K maps certain elements $\alpha \in L$ to $f(\zeta_q)\alpha$. On studying the prime factorization of α he infers, back to K, a decomposition

$$(N_{L/K}(\alpha)) = R^b \prod_{\sigma \in \Delta} \sigma^{-1}(Q)^{r_\sigma}, \qquad (*)$$

valid under certain conditions, where R and Q are ideals of K, Q a prime above q, and b is a divisor of $q-1$. The exponents r_σ have a deep connection to the units of K. Here a crucial role is played by a local-global theorem depending on Chebotarev's density theorem.

From (*) it follows that if R^b is principal then $\sum_{\sigma\in\Delta} r_\sigma \sigma^{-1}$ annihilates the ideal class $\mathcal{C}$ containing Q. By use of the properties of r_σ this implies the following results: If $b = p^n$ is the exponent of $(A)_p$ and if δ satisfies, for all $\sigma \in \Delta$, a congruence $\sigma(\delta) \equiv \delta^{c_\sigma} \bmod E^{p^n}$ with $p \nmid c_\sigma$, then $\sum_{\sigma\in\Delta} c_\sigma \sigma^{-1}$, multiplied by a certain integer $2d$, annihilates all classes $\mathcal{C} \in (A)_p$ containing an ideal Q. If fact, d depends on $|\delta|$ in a simple way.

Suppose that $p \nmid [K:\mathbb{Q}]$ and χ is a non-trivial p-adic valued character of Δ, with the corresponding idempotent e_χ of $\mathbb{Z}[\Delta]$. Then, by taking $c_\sigma = \chi(\sigma)$ one proves that $e_\chi(A)_p$ is annihilated by the exact exponent p^a of $e_\chi(W)_p$. As corollary it follows that if K is a subfield of $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$, then every annihilator of $(W)_p$ annihilates $(A)_p$. Finally, a generalization of the above ideas leads to the beautiful theorem:

If $p \nmid [K:\mathbb{Q}]$ and if $\theta \in \mathbb{Z}[\Delta]$ annhilates $(W)_p$, then 2θ annhilates $(A)_p$.

Reviewer: Tauno Metsänkylä (Turku)

MSC:

11R18 Cyclotomic extensions
11R29 Class numbers, class groups, discriminants
11G16 Elliptic and modular units
11R27 Units and factorization

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ideal class group; unit group; real abelian field; annihilators of the p-class group; circular units; local-global theorem; density

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