Gouveia, João; Pashkovich, Kanstanstin; Robinson, Richard Z.; Thomas, Rekha R.
Four-dimensional polytopes of minimum positive semidefinite rank. (English) Zbl 1360.52006
J. Comb. Theory, Ser. A 145, 184-226 (2017).

Summary: The positive semidefinite (psd) rank of a polytope is the size of the smallest psd cone that admits an affine slice that projects linearly onto the polytope. The psd rank of a d-polytope is at least $d+1$, and when equality holds we say that the polytope is psd-minimal. In this paper we develop new tools for the study of psd-minimality and use them to give a complete classification of psd-minimal 4-polytopes. The main tools introduced are trinomial obstructions, a new algebraic obstruction for psd-minimality, and the slack ideal of a polytope, which encodes the space of realizations of a polytope up to projective equivalence. Our central result is that there are 31 combinatorial classes of psd-minimal 4-polytopes. We provide combinatorial information and an explicit psd-minimal realization in each class. For 11 of these classes, every polytope in them is psd-minimal, and these are precisely the combinatorial classes of the known projectively unique 4-polytopes. We give a complete characterization of psd-minimality in the remaining classes, encountering in the process counterexamples to some open conjectures.

MSC:

52B11 n-dimensional polytopes
52B05 Combinatorial properties of polytopes and polyhedra (number of faces, shortest paths, etc.)

Keywords:

polytopes; positive semidefinite (psd) rank; psd-minimal; slack matrix; slack ideal

Software:

2L_enum; Macaulay2; SageMath

Full Text: DOI

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