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A sharp Balian-Low uncertainty principle for shift-invariant spaces. (English) Zbl 1391.42036
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The authors prove a sharp version of the Balian-Low theorem for the generators of finitely generated shift-invariant spaces. For singly generated shift-invariant spaces their main results take the following form: 1.(Corollary 1.4) Fix lattices $\Lambda, \Gamma \subset \mathbb{R}^d$ with $\Lambda \subset \Gamma$ and with the index of the lattice Λ in Γ greater than 1 ($[\Gamma : \Lambda] > 1$). Suppose $f \in L^2(\mathbb{R}^d)$, $\|f\|_2 \neq 0$, and $\mathcal{T}^\Lambda(f) = \{f(\cdot - \lambda) : \lambda \in \Lambda\}$ forms a frame for the closed linear span $V^\Lambda(f)$ of $\mathcal{T}^\Lambda(f)$ in $L^2(\mathbb{R}^d)$. If $V^\Lambda(f)$ is Γ -invariant, then $\int_{L^2(\mathbb{R}^d)} |x||f(x)|^2 dx = \infty$. 2.(Corollary 1.6) Fix a lattice $\Lambda \subset \mathbb{R}^d$. Suppose $f \in L^2(\mathbb{R}^d)$ with $\|f\|_2 \neq 0$. If $\mathcal{T}^\Lambda(f)$ is a frame for $V^\Lambda(f)$, but is not a Riesz basis for $V^\Lambda(f)$, then $\int_{L^2(\mathbb{R}^d)} |x||f(x)|^2 dx = \infty$. The main results provide an absolutely sharp improvement of the best previously existing results in the literature in this direction.

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