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Cyclotomic polynomials at roots of unity. (English) Zbl 1435.11060
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Let Φ_n be the nth cyclotomic polynomial and let ξ_m be an arbitrary primitive mth root of unity ($e^{2\pi ia/m}$), $(a,m)=1$). The paper under review evaluates $\Phi_n(\xi_m)$ for $1\le m\le 6$ and all n. These evaluations splinter into quite a few cases with some of the cases yielding rather complicated values, and we shall not state them here explicitly. To give a reader of this review a glimpse of what happens, we just mention that even the simplest case of $\xi_1=1$ leads to three subcases: of course $\Phi_1(1)=0$, and for $n\ge 2$, $\Phi_n(1)=1$, except for when n is a prime power in which case $\Phi_{p^\nu}(1)=p$. This and a closely related evaluation of $\Phi_n(\xi_2)=\Phi_n(-1)$ are folklore background to this problem.

An earlier work of *K. Motose* [Bull. Fac. Sci. Technol., Hirosaki Univ. 9, No. 1, 15–27 (2006; Zbl 1193.11115)] gave evaluations of $\Phi_n(\xi_m)$ for $m=3,4$ and 6, except that it was blemished by some error. The paper under review not only corrects the record, but obtains these evaluations by a different and more efficient technique.

As we shall explain presently, the case $m=5$ is considerably more interesting and important. The authors' work on this case utilizes a new general identity of independent interest which we explain next. For this identity we assume that $n,m>1$ are coprime. Then we have

$$\Phi_n(\xi_m)=\exp\Big(\frac{1}{\varphi(m)}\sum_{\chi}C_\chi(\xi_m)D_\chi(n)\Big),$$

where the summation is over Dirichlet characters χ modulo m,

$$C_\chi(\xi_m)=\sum_{(a,m)=1}\bar\chi(a)\log(1-\xi_m^a)\quad\text{and}\quad D_\chi(n)=\chi(n)\prod_{p|n}(1-\bar\chi(p)).$$

The basic idea here is as follows. From the well-known Möbius inversion formula $\Phi_n(z)=\prod_{d|n}(1-z^d)^{\mu(n/d)}$, we get

$$\Phi_n(\xi_m)=\exp\Big(\sum_{d|n}\mu(n/d)\log(1-\xi_m^d)\Big).$$

The authors observe that $\log(1-\xi_m^d)$, as a function of d with $(d,m)=1$, has a representation

$$\log(1-\xi_m^d)=\frac{1}{\varphi(m)}\sum_{\chi}C_\chi(\xi_m)\chi(d),$$

with the coefficients C_χ computed in the standard manner. The rest of the derivation is then routine.

Utilizing the forgoing identity, the authors succeed in evaluating $\Phi_n(\xi_5)$. As we already mentioned, this case is special, as first realized by *R. C. Vaughan* [Mich. Math. J. 21, 289–295 (1975; Zbl 0304.10008)]. Let A_n denote the maximum absolute value of the coefficients of Φ_n. Since degree of Φ_n is $\varphi(n)$, where φ is the Euler totient, the inequality

$$A_n\ge\frac{\max_{|z|=1}|\Phi_n(z)|}{\varphi(n)+1}$$

is immediate. Vaughan conceived of the utility of using ξ_5 for manufacturing arbitrary large integers n for which $|\Phi_n(\xi_5)|$ were particularly large, as a function of n. More specifically, he proved that there exist arbitrarily large integers n for which

$$\log\log A_n>\log 2\frac{\log n}{\log\log n}.$$

It follows from an earlier result of *P. T. Bateman* [Bull. Am. Math. Soc. 55, 1180–1181 (1949; Zbl 0035.31102)] that the constant $\log 2$ is the best possible in this inequality. Thus the evaluation of $\Phi_n(\xi_5)$ given in the paper under review may be viewed as a refinement of the aforementioned work of Vaughan, and a new proof of the last inequality.

Reviewer: Gennady Bachman (Las Vegas)

MSC:

11C08 Polynomials in number theory

11R09 Polynomials (irreducibility, etc.)

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cyclotomic polynomial; cyclotomic coefficient; value at root of unity; resultant; Vaughan's theorem

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