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Local duality in algebra and topology. (English) Zbl 1403.55008

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Local duality is a relation between local cohomology and local homology. Examples include Grothendieck's local duality, and its generalisation by Greenlees and May. This paper constructs a simple abstract framework for local duality statements in general. The named examples are then deducible from this framework and the authors consider a number of further examples of local duality that fit into their theory. These examples include categories of comodules over Hopf algebroids, chromatic homotopy theory, equivariant homotopy theory, motivic homotopy theory and local duality for schemes.

The general results are phrased in the context of a symmetric monoidal triangulated category $\mathcal{C}$ with some underlying homotopy theory, such as a pre-triangulated dg-category, stable model category or stable ∞-category. These categories are called stable categories.

Let $\mathcal{K}$ denote a set of compact objects of $\mathcal{C}$. From this set, three more stable categories are defined. Let $\mathcal{C}^{\mathcal{K}\text{-tors}}$ be the localising ideal generated by $\mathcal{K}$: the full triangulated subcategory of $\mathcal{C}$ that contains $\mathcal{K}$, is closed under retracts, desuspensions and filtered colimits and tensor products with objects of $\mathcal{C}$. Let $\mathcal{C}^{\mathcal{K}\text{-loc}}$ be those objects of $\mathcal{C}$ which admit no maps from objects of $\mathcal{C}^{\mathcal{K}\text{-tors}}$. Let $\mathcal{C}^{\mathcal{K}\text{-comp}}$ be those objects of $\mathcal{C}$ which admit no maps from objects of $\mathcal{C}^{\mathcal{K}\text{-loc}}$.

The inclusion i_{tors} of $\mathcal{C}^{\mathcal{K}\text{-tors}}$ into $\mathcal{C}$ has a right adjoint Γ, called the torsion functor, and may be thought of as a local cohomology functor. Similarly, the inclusion i_{loc} of $\mathcal{C}^{\mathcal{K}\text{-loc}}$ into $\mathcal{C}$ has a left adjoint L and the inclusion i_{comp} of $\mathcal{C}^{\mathcal{K}\text{-comp}}$ into $\mathcal{C}$ has a left adjoint Λ. The functor Λ is called the completion functor and may be thought of as local homology. The functors Γ and L are smashing and the inclusions and their adjoints induce an equivalence of stable categories

$$\mathcal{C}^{\mathcal{K}\text{-loc}} \simeq \mathcal{C}^{\mathcal{K}\text{-comp}}.$$

The duality statement is the adjunction

$$\underline{\text{Hom}}(i_{\text{tors}}\Gamma X, Y) \simeq \underline{\text{Hom}}(X, i_{\text{comp}}\Lambda Y)$$

where $\underline{\text{Hom}}$ denotes the internal function object of $\mathcal{C}$. The pullback of the functors

$$L \longrightarrow L\Lambda \longleftarrow \Lambda$$

is the identity, giving a fracture square analogous to the Hasse squares in stable homotopy theory.

The rest of the paper then examines various examples of where this formalism applies as above. Of particular note is the new duality result on comodules over a Hopf algebroid, which is then extended to quasi-coherent sheaves on a presentable algebraic stack. There is a substantial section on chromatic homotopy theory, considering three settings: p-local spectra, $E(n)$-local spectra and $E(n)$-modules. The paper demonstrates how these topological duality statements are related to algebraic duality statements of comodules: taking R to be BP, $E(n)$ or the sphere spectrum (respectively) the homology functor R_* will send spectra to R_*R-comodules in a manner compatible with the duality constructions.

Reviewer: David Barnes (Belfast)

MSC:

55P60 Localization and completion in homotopy theory
13D45 Local cohomology and commutative rings
14B15 Local cohomology and algebraic geometry
55U35 Abstract and axiomatic homotopy theory in algebraic topology
55U30 Duality in applied homological algebra and category theory (aspects of algebraic topology)

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