Hodge, W. V. D.

The theory and applications of harmonic integrals. Reissued, with a foreword by Sir Michael Atiyah. (English) Zbl 0693.14002

Cambridge Mathematical Library. Cambridge: Cambridge University Press. xiii, 284 p. (1989).

Hindsight distorts all our judgement of past achievements, but especially so in mathematics. What was once inconceivable becomes rapidly convincing, possible, proved, standard and even trivial. It is the daily experience of students of mathematics to go through this process, leaping hurdles which seemed insurmountable and then forgetting they ever existed. To travel back 50 years in time, and see one of the standard tools of global geometry at the moment of its birth is therefore a sobering experience and one which forces us to consider our present day achievements in a new light. That is what the book under review compels us to do, as it requires us to view Hodge theory in the light of its times.

The book itself was termed by Hermann Weyl in 1954 as "one of the great landmarks in the history of science in the present century". It brought together the subjects of algebraic geometry, topology, differential geometry and differential equations to produce powerful results in a manner which was to a certain extent controversial at the time. [A good companion for reading the book nowadays is Atiyah's obituary for Hodge; cf. *M. F. Atiyah*, Bull. Lond. Math. Soc. 9, 99-118 (1977; Zbl 0343.01010), where the reader will learn about the initial dispute with Lefschetz over some of the results.] The technical problems which flaw the existence proof, correct proofs of which were given by Weyl and Kodaira, have never obscured the impact of the work which revealed new truths and harmonized old ones and fully justified Weyl's tribute.

Hodge's idea, motivated by the "duality" between the real and imaginary parts of a holomorphic differential on a Riemann surface was to introduce for each p-form a dual (n-p)-form on an n-dimensional manifold. The form is harmonic if it and its dual are closed, and the main theorem is that there is a unique harmonic form with prescribed periods. The applications are to projective algebraic varieties where the Riemannian metric needed to define the notion of duality is induced from the standard metric on projective space, and to the study of invariant forms on compact Lie groups. These results are of course common knowledge among geometers today, but we have the benefit of names and concepts to give a perspective to the theorem - de Rham cohomology, Kähler metrics, parametrices for elliptic operators and so forth. Each one is a theory in its own right, and we nowadays manipulate them as tools to achieve our ends. In Hodge's day the work had to be done at ground level using the basic building tricks of each discipline, many of which were new or not well-known.

The book may well have been difficult to read when it was first published and it remains so, though for good reasons - it is as if one were to be asked to give a graduate course with no prerequisites. Even now it is not easy to find a self-contained account of Hodge theory in the literature, one of the few appearing only in 1971 in *F. W. Warner*'s book ["Foundations of differentiable manifolds and Lie groups" (1971; Zbl 0241.58001; reprint 1983)]. This is a good measure of the difficulties which faced Hodge, and which still face the uninitiated reader. Those who know the subject, however, will recognize the significant advances gradually emerging in the text.

Reading it, one is occasionally struck by the order in which results are proven and the somewhat altered emphasis between the original treatment and the accepted modern way of dealing with things. The duality of forms is fundamental to Hodge, the Laplacian being very much in the background. The author also apologizes for treating "only" the biholomorphic rather than birational geometry of projective varieties, the latter clearly being the accepted notion of equivalence in the algebraic geometry of the time. The decomposition of cohomology in terms of primitive forms, depending essentially on what we now call the Kähler structure, is studied in depth well before the (p,q)-decomposition which we now think of as being more basic. Above all, one is conscious of the notation - the plethora of suffixes, local computations, geometrical entities defined by "laws", and the subscript "x" defining exterior derivative. The separate distillations of cohomology theory, abstract algebraic geometry, and global analysis which grew out of this rich ferment owe a great deal to the choice of an appropriate, suggestive and manipulable formalism. For us, only Hodge's star * remains, and very convenient it is too.

There are many lessons to be learned from reading this reissued volume. One is the continuing relevance of Hodge's work, usefully summarized by *Atiyah* in a foreword. Another one is the way it forces the reader to reevaluate his or her conception of the manner in which new mathematics is born. One may think of some subjects which nowadays bring together different disciplines and give positive new results, are recognized as being powerful and yet are an ungainly mixture of elementary computations and long and difficult theorems. If hindsight and this book tell us about Hodge theory and its history, they may also tell us something about the future of, for example, quantum groups, integrable systems and knot theory.

Reviewer: N.J.Hitchin

MSC:

14C30	Transcendental methods, Hodge theory (algebro-geometric aspects)	Cited in **7** Documents
32G20	Period matrices, variation of Hodge structure; degenerations	
14-03	History of algebraic geometry	
01A60	History of mathematics in the 20th century	
01A75	Collected or selected works; reprintings or translations of classics	

Keywords:

history of Hodge theory; harmonic integrals; period matrices; duality; de Rham cohomology; Kähler metrics