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Hurewicz images of real bordism theory and real Johnson-Wilson theories. (English)

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Let $\mathbb{S}$ be the sphere spectrum and let $MU_{\mathbb{R}}$ be the real cobordism spectrum in the sense of *P. S. Landweber* [Bull. Am. Math. Soc. 74, 271–274 (1968; Zbl 0181.26801)]. In [Ann. Math. (2) 184, No. 1, 1-262 (2016; Zbl 1366.55007)], *M. A. Hill* et al. showed that the Kervaire invariant elements $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$ do not exist for $j \geq 7$. It is well known that the spectrum $MU_{\mathbb{R}}$ splits as a wedge of suspensions of $BP_{\mathbb{R}}$, a real analogue of the Brown-Peterson spectrum, in [*S. Araki*, Jpn. J. Math., New Ser. 5, 403–430 (1979; Zbl 0443.55003)]; that is,

$$MU_{\mathbb{R}} = \bigvee_{m_i} \Sigma^{m_i(1+\rho)} BP_{\mathbb{R}},$$

where ρ is the non-trivial one dimensional real representation of C_2. It is also well known that the Hopf elements are represented by the elements $h_i \in \mathrm{Ext}^{1,2^i}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ on the E_2-page of the classical Adams spectral sequence at the prime 2. *W. Browder* [Ann. Math. (2) 90, 157–186 (1969; Zbl 0198.28501)] showed that the Kervaire classes $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$ are represented by the elements $h_j^2 \in \mathrm{Ext}^{2,2^{j+1}}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ on the E_2-page if they exist. *W.-H. Lin* [Topology Appl. 155, No. 5, 459–496 (2008; Zbl 1140.55012)] proved that there is a family $\{g_k \mid k \geq 1\}$ of indecomposable elements with $g_k \in \mathrm{Ext}^{4,2^{k+2}+2^{k+3}}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$. The elements g_1 and g_2 of the family survive the Adams spectral sequence to become $\bar{\kappa} \in \pi_{20}\mathbb{S}$ and $\bar{\kappa}_2 \in \pi_{44}\mathbb{S}$, respectively. The $\bar{\kappa}$*-family* consists of the homotopy classes detected by the surviving g_k-family.

In this paper, the authors show that the Hopf elements, the Kervaire classes and the $\bar{\kappa}$-family are detected by the Hurewicz maps $\pi_*\mathbb{S} \to \pi_* MU_{\mathbb{R}}^{C_2}$ and $\pi_*\mathbb{S} \to \pi_* BP_{\mathbb{R}}^{C_2}$, and that the G-fixed point of $MU^{((G))}$ detects the Hopf elements, the Kervaire classes and the $\bar{\kappa}$-family for any finite group G containing C_2. Moreover, the authors show that the images of the elements $\{h_i \mid i \geq 1\}$, $\{h_j^2 \mid j \geq 1\}$ and $\{g_k \mid k \geq 1\}$ on the E_2-page of the classical Adams spectral sequence of the sphere spectrum are nonzero on the E_2-page of the C_2-equivariant Adams spectral sequence of $BP_{\mathbb{R}}$. They also prove that the integer-graded C_2-equivariant May spectral sequence of $BP_{\mathbb{R}}$ is isomorphic to the associated-graded slice spectral sequence of $BP_{\mathbb{R}}$, and that a subset of those families is detected by the C_2-fixed points of Real Johnson-Wilson theory $E\mathbb{R}(n)$ depending on n. More precisely, (1) if the element $h_i \in \mathrm{Ext}^{1,2^i}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ or $h_j^2 \in \mathrm{Ext}^{2,2^{j+1}}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E\mathbb{R}(n)^{C_2}$ is nonzero for $1 \leq i, j \leq n$, and (2) if the element $g_k \in \mathrm{Ext}^{4,2^{k+2}+2^{k+3}}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E\mathbb{R}(n)^{C_2}$ is nonzero for $1 \leq k \leq n-1$.

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MSC:

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55T20 Eilenberg-Moore spectral sequences

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