Montalbán, Antonio; Walsh, James

On the inevitability of the consistency operator. (English) Zbl 1456.03091

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The paper is concerned with recursive monotone functions on the Lindenbaum-Tarski algebra of the elementary arithmetic EA. Denote the equivalence class of a sentence φ, modulo EA-provability, by $[\varphi]$. Write $[\psi] \leq [\varphi]$ when $\mathrm{EA} \vdash \psi \to \varphi$. A function f on sentences is *monotone* if $f([\psi]) \leq f([\varphi])$ for every ψ and φ with $[\psi] \leq [\varphi]$. Note that then f is also *extensional* in the sense that $f([\psi]) = f([\varphi])$ when $[\psi] = [\varphi]$.

V. Yu. Shavrukov and *A. Visser* proved in [Notre Dame J. Formal Logic 55, No. 4, 569–582 (2014; Zbl 1339.03056)] that there exists a recursive extensional function on the Lindenbaum-Tarski algebra of Peano's arithmetic such that it lies strictly between the identity function and the mapping $\varphi \mapsto \varphi \wedge \mathrm{Con}(\varphi)$. The main result of the present paper is that there is no recursive monotone function on the Lindenbaum-Tarski algebra of Con that lies strictly between the identity function and the mapping $\varphi \mapsto \varphi \wedge \mathrm{Con}(\varphi)$.

This also negatively answers a question of Shavrukov and Visser in [loc. cit.] which asked if there exists a recursive binary function that is monotonic in both its coordinates and satisfies (i) $[\psi] < g([\psi], [\varphi]) < [\varphi]$ if $[\psi] < [\varphi]$, and (ii) $g([\psi], [\theta]) = g([\varphi], [\theta])$ and $g([\theta], [\psi]) = g([\theta], [\varphi])$ if $[\psi] = [\varphi]$.

The authors also generalize their results "to iterates of consistency into the effective transfinite". They leave open three interesting questions in the paper, one of which is the following: if f is a recursive monotone Π^0_1(-valued) function such that for every φ we have $[\varphi \wedge \mathrm{Con}^\alpha(\varphi)] \leq [f(\varphi)]$ for some fixed ordinal α, must there exist some $\beta \leq \alpha$ and some true sentence θ such that $[\theta \wedge f(\theta)] = [\theta \wedge \mathrm{Con}^\beta(\theta)]$?

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MSC:

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