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Spanning homogeneous vector bundles. (English) Zbl 0705.14014

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Let B be a Borel subgroup generated in the complex Lie group G by the negative roots of G. Let G_α denote the rank one simple subgroup of G generated by a positive root α, and let B_α be the intersection of G_α with B. Then $B_\alpha = T_\alpha U_{-\alpha}$ where T_α is a maximal torus of G_α and $U_{-\alpha}$ the unipotent subgroup generated by -α. Let E be a B-module. Considered as a $U_{-\alpha}$-module, E extends to a G_α-module and decomposes uniquely (up to rearrangement) $E = E_1 \oplus \cdots \oplus E_k$ where $E_i = m_{i,\alpha}\lambda_\alpha | G_\alpha$ is the G_α-module induced from a nonnegative multiple of the fundamental dominant weight λ_α. Each summand E_i is invariant under T_α with highest weight $t_{i,\alpha}\lambda_\alpha$, $1 \leq i \leq k$. Thus, as a B_α-module, $E_i = m_{i,\alpha}\lambda_\alpha|^{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha$ with $n_{i,\alpha} = t_{i,\alpha} - m_{i,\alpha}$. The elements of the sequence of integers $n_{i,\alpha}$, $i = 1, ..., k$, are called the α-indices of E.

By $E|^G$ we denote the induced G-module of all B-equivariant algebraic maps $G \to E$, and the evaluation map ϵ is the function $v \mapsto v(1):\ E|^G \to E$. For a B-module E, the associated homogeneous vector bundle $\mathcal{E} = G \times_B E$ is spanned by the global sections if and only if ϵ is surjective. For an element w in the Weyl group with reduced expression $s_1 \cdots s_{i_n}$ let X_w denote the closure of BwB in G/B. Call $\mathcal{E}_w$ the restriction of $\mathcal{E}$ to X.

Theorem. The vector bundle $\mathcal{E}_w$ is spanned by global sections if and only if the α-indices of E are nonnegative for all simple roots α_j, $j = i_1, ..., i_n$, corresponding to the sequence of reflections s_k.

Corollary. A homogeneous vector bundle $\mathcal{E}$ is spanned by global sections if and only if the α-indices of E are nonnegative for all simple roots α.

It is further shown that $\mathcal{E}$ is spanned by global sections if and only if for some n, the n-th symmetric power $S^n(\mathcal{E})$ is spanned by global sections if and only if ξ_E^n is spanned by global sections for some n, where ξ_E is the tautological line bundle over the projectivization $\mathbb{P}(\mathcal{E})$ of $\mathcal{E}$ whose restriction to the fiber $\mathbb{P}(E)$ is $\mathcal{O}(1)$.

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