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Relèvement des caractères d'un groupe endoscopique pour le changement de base $\mathbb{C}/\mathbb{R}$. (Lifting of characters of an endoscopic group for the base change $\mathbb{C}/\mathbb{R}$). (French)

Zbl 0708.22004
Orbites unipotentes et représentations. II: Groupes p-adiques et réels, Astérisque 171-172, 163-194 (1989).

[For the entire collection see Zbl 0694.00012.]

Let G be a connected reductive algebraic group defined over $\mathbb{R}$. Then the "functor" $\mathbb{R} \mapsto G(\mathbb{C})$ and $\mathbb{C} \mapsto G(\mathbb{C}) \times G(\mathbb{C})$ is (part of) the points functor of another connected reductive algebraic group defined over $\mathbb{R}$, notated $\tilde{G}$ and called the result, applied to G, of the restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$, or, base change. *L. Clozel* [Ann. Sci. Éc. Norm. Supér., IV. Sér. 15, 45-115 (1982; Zbl 0516.22010)] had established a relation between representations π of $G(\mathbb{R})$ and Π of $\tilde{G}(\mathbb{R}) = G(\mathbb{C})$. If Π is equivalent to $\Pi \circ \sigma$ where σ is the Galois action on $G(\mathbb{C})$ then we may "twist" the trace of Π by an intertwining operator A_σ that implements this equivalence. Denote the twisted character (which depends on the choice of A_σ) by χ^σ_Π, and the ordinary character of π by χ_π. The so-called norm "map" N given on $G(\mathbb{C})$ by $g \mapsto gg^\sigma$ does not, of course, necessarily land in $G(\mathbb{R})$ but up to stable conjugacy (i.e. conjugacy in $G(\mathbb{R})$ by elements of $G(\mathbb{C})$ it does, so let Ng denote the stable conjugacy class obtained. Then for π the direct sum of all irreducible representations L-indistinguishable to a given tempered one, Clozel showed that there existed a unique irreducible tempered representation Π of $\tilde{G}$ with $\chi^\sigma_\Pi(g) = \pm\chi_\pi(Ng)$, a result first proved by Shintani for GL_2 (but for p-adic and finite fields as well), at least if A_σ is normalized. Furthermore, Π has the Langlands parameter obtained from that of π by the functoriality of base change.

The appearance of L-indistinguishability suggests that endoscopy and a matching of unstable combinations of twisted orbital integrals for $\tilde{G}$ with stable orbital integrals for certain reductive quasisplit groups H of lower dimension should be possible. This was accomplished by *D. Shelstad* [Pac. J. Math. 110, 397-416 (1984; Zbl 0488.22033)]. Dual to the matching of orbital integrals is a map Tran on (certain) stably-invariant distributions of $H(\mathbb{R})$ to σ-twisted- invariant distributions on $\tilde{G}(\mathbb{R})$, easily calculated explicitly, cf. lemme 3-3-1 of the paper under review, whose purpose is to establish the generalization of Clozel's result and show that if π is as above but for $H(\mathbb{R})$, then

$$Tran^{\tilde{G}}_H(\chi_\pi)(g) = \pm\chi^\sigma_\Pi(g),$$

where the Langlands parameter of Π is obtained from that of π by the functoriality of twisted endoscopy. That is, whenever H is one of Shelstad's twisted endoscopic groups for $\tilde{G}$ and an allowed embedding $\tilde{\xi}$: ${}^LH \to^L \tilde{G}$ exists, then if $\phi : W_\mathbb{R} \to^L H$ is the Langlands parameter for the tempered L-packet of (any of the summands of) π (and for simplicity, say $\tilde{\xi}$ preserves temperedness), we have that $\tilde{\xi} \circ \phi : W_\mathbb{R} \to^L \tilde{G}$ is the Langlands parameter of the tempered irreducible representation Π of $\tilde{G}(\mathbb{R})$.

The proof follows the approach of the author's earlier work on the orbit method and Kirillov's character formula for disconnected groups: indeed, in [J. Funct. Anal. 70, 1-79 (1987; Zbl 0622.22009)] he had already shown its applicability to Clozel's results. This character formula is only valid for tempered representations and only on the elliptic regular elements. Fortunately, for tempereds, the character, or, as Clozel showed, its twisted character (which is the same as the character of the extension of Π to $G(\mathbb{C}) \times \{1, \sigma\}$ by A_σ, on the connected component of σ), is determined by its values on the elliptic set (respectively, the elements whose norms are elliptic).

Base change of a special family of non-tempered unitary representations has been treated by the reviewer [Math. Ann. 287, 467-493 (1990; Zbl 0672.22016)] by algebraic methods. The same methods show, for the same family of representations, that the functoriality of twisted endoscopy also holds, by reducing to the tempered case and using the author's results. The more general case is still an open problem.

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MSC:

22E45 Representations of Lie and linear algebraic groups over real fields: analytic methods

Cited in **4** Documents

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connected reductive algebraic group; base change; Galois action; trace; intertwining operator; twisted character; stable conjugacy class; irreducible representations; irreducible tempered representation; functoriality of base change; L-indistinguishability; endoscopy; twisted orbital integrals; stably-invariant distributions; Langlands parameter; twisted endoscopic groups; tempered L-packet; orbit method; Kirillov's character formula; unitary representations