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Stratification and duality for homotopical groups. (English) Zbl 1426.55017
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Let G be a finite group. Some of the most celebrated results in group cohomology are the F -isomorphism theorem of Quillen and the study of the spectrum of the cohomology ring of G with coefficients in a field k whose characteristic divides the order of G . There have been generalizations of these results to other instances like p -compact groups and p -local finite groups. The main objective of this paper is to generalize these (and many related) results to p -local compact groups. Much of the work is devoted to establishing the technical tools to achieve these results. We provide some examples. A p -local compact group $\mathcal{G} = (S, \mathcal{F})$ satisfies Choinard's theorem if induction and coinduction along the morphism induced by restriction

$$C^*(B\mathcal{G}, k) \rightarrow \prod_{\mathcal{E}(\mathcal{G})} C^*(BE, k),$$

is conservative, where $\mathcal{E}(\mathcal{G})$ denotes a set of representatives of \mathcal{F} -conjugacy classes of elementary abelian subgroups of S and $C^*(B(-), k)$ is the spectrum of k -valued cochains on $B(-)$. The authors prove that connected p -local groups satisfy Choinard's theorem. They also prove the generalization of Quillen's F -isomorphism theorem to these groups:

Theorem. Let $\mathcal{G} = (S, \mathcal{F})$ be a p -local compact group, the restriction to elementary abelian subgroups of S induces an F -isomorphism

$$H^*(B\mathcal{G}, \mathbb{F}_p) \rightarrow \lim_{\mathcal{F}^e} H^*(BE, \mathbb{F}_p),$$

where \mathbb{F}^e denotes the full subcategory of \mathbb{F} consisting elementary abelian p -groups.

They also prove that p -local compact groups satisfy Quillen's stratification.

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MSC:

[55R35](#) Classifying spaces of groups and H -spaces in algebraic topology
[20J05](#) Homological methods in group theory
[13D45](#) Local cohomology and commutative rings
[55P42](#) Stable homotopy theory, spectra

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