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Optimal Littlewood-Offord inequalities in groups.  (English) [Zbl 07134837]
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The paper under review is to prove several Littlewood-Offord type inequalities in arbitrary groups. J. E. Littlewood and A. C. Offord [Mat. Sb., Nov. Ser. 12(54), 277–286 (1943; Zbl 0061.01801)] proved a bound for the probability that a sum of random signs with non-zero weights hits a point which is asymptotically optimal up to a logarithmic factor. Let $V_n = \{g_1, \ldots, g_n\}$ be a set of non-identity elements of an arbitrary group $G$, and $X_i$ be an independent random variable that are each distributed on a two-element set $\{g_i, g_i^{-1}\}$. Define

$$\rho(V_n) = \sup_{g \in G} P(X_1 \cdots X_n = g).$$

If $G = (R, +)$ is an additive group of real numbers, then $X_i = \varepsilon_i$ on $\{\pm 1\}$ for $g_i = 1$ for all $i$. The Littlewood-Offord result shows that $\rho(V_n) = O(n^{-1/2})$. Using Sperner’s theorem from finite set combinatorics, Erdős proved that $\rho(V_n) \leq \left(\frac{\log n}{n}\right)^{1/2}$, and this bound is optimal and can be achieved. J. R. Griggs [Bull. Am. Math. Soc., New Ser. 28, No. 2, 329–333 (1993; Zbl 0797.11079)] showed a similar optimal bound for finite cyclic group $\mathbb{Z}_n$. For the matrix group $GL_d(C)$ with $m, d, n \geq 2$, P. H. Tiep and V. H. Vu [Adv. Math. 302, 1233–1250 (2016; Zbl 1346.05295)] proved $\rho(V_n) \leq 141 \max\{1, \frac{1}{m}, \frac{1}{\sqrt{n}}\}$.

The authors of the paper under review show that for any $A \subset G$ with odd or infinite order

$$P(X_1 \cdots X_n \in A) \leq P(\varepsilon_1 + \cdots + \varepsilon_n \in (-k,k])$$

i.e., Theorem 1 holds and is optimal in the case that $G$ contains an element of order $m$. For $g_i$ of order $\geq m \geq 2$, $\rho(V_n) \leq \frac{2}{m} + \sqrt{\frac{2}{n \pi}} \leq 3 \max\{1, \frac{1}{m}, \frac{1}{\sqrt{n}}\}$. Theorem 1 is proved in Section 3 with a simple group theoretic lemma on countings. The simple random walk on $\mathbb{Z}_m$ for odd $m$ converges to the uniform distribution on $\mathbb{Z}_m$ and

$$\rho(V_n) = P(\varepsilon_1 + \cdots + \varepsilon_n \in (-1,1]) = (1 + o(1))\sqrt{\frac{2}{n \pi}}.$$

Theorem 3 shows that $\sup_{g \in G} P(X_i = g) \leq \frac{1}{2}$. It is well-known that the distribution of the sum of independent uniform random distribution is asymptotic uniform in $\mathbb{Z}_m$ and $P(X_1 \cdots X_n = g) \leq \frac{1}{m} + o(1)$. The proofs follow in the same spirit of D. J. Kleitman [Adv. Math. 5, 155–157 (1970; Zbl 0195.40703)] in his solution to the Littlewood-Offord problem in arbitrary dimensions. Section 2 presents an open problem in a conjecture for elements of $G$ with even orders. The authors hope that Theorem 1 and Theorem 3 hold, in order to complete this problem.

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MSC:

20-XX Group theory and generalizations
05D40 Probabilistic methods in extremal combinatorics, including polynomial methods (combinatorial Nullstellensatz, etc.)
60B15 Probability measures on groups or semigroups, Fourier transforms, factorization

Keywords:

Littlewood-Offord problem; finite group; matrix group; uniform distribution; discrete random variables

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