Iwasawa L-functions for multiplicative abelian varieties. (English) [Zbl 0716.14008]
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Iwasawa L-functions for abelian varieties with multiplicative reductions are studied, extending some results proved by B. Mazur in Invent. Math. 18, 183-266 (1972; Zbl 0245.14048)

Let \( p \neq 2 \) be a prime, \( \Gamma = \mathbb{Z}_p \) (as an additive topological group) with a generator \( \gamma \), and \( \Lambda := \lim_{\longrightarrow} \mathbb{Z}_p \Gamma/p^n \Gamma \). Then the map which sends \( T \) to \( \gamma^{-1} T \) induces an isomorphism from \( \mathbb{Z}_p \Gamma/\mu(T) \) to \( \Lambda \). The Iwasawa L-function for an elliptic curve was defined as the characteristic polynomial of the p-Selmer group of the curve. To generalize this definition to abelian varieties, one needs “good” \( \Lambda \)-modules which are finitely generated modules \( M \) over \( \Lambda \). Such a module is quasi-isomorphic to the direct sum \( N^j \oplus \mathbb{Z}/p^j \mathbb{Z}[[T]] \oplus (\otimes_j \mathbb{Z}_p[[T]](F_j)p^n) \) where \( p \) is the free rank of \( M, F_j \) is an irreducible distinguished polynomial for each \( j \). The invariants \( (\rho, \mu, \{E_j\}) \) determine \( M \) completely up to quasi-isomorphism (i.e., up to finite kernel and cokernel). The \( \mu \)-invariant of \( M \) is \( \mu := \sum \mu_i \), the characteristic polynomial of \( M \) is \( F_M(T) := p^\mu \prod_j (F_j(T))^{\mu_j} \) and \( f_M(T) \) is the polynomial satisfying \( f_M(T+1) = F_M(T) \).

Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \). Let \( A/K \) be an abelian variety defined over \( K \), \( \hat{A} \) its Néron model over \( \mathcal{O}_K \), \( \hat{A}_p \) the dual abelian variety of \( A \), \( A^0 \) the connected component of \( A \) and \( A_{\infty} := \cup_p A_p^0 \). Let \( \Phi \) be defined by the short exact sequence \( 0 \rightarrow A^0 \rightarrow A \rightarrow \Phi \rightarrow 0 \). Let \( L/K \) be a \( \Gamma \)-extension of \( K \), \( T \) the set of all primes in \( K \) ramifying in \( L \), \( \log_p \rho \), a p-adic logarithm of \( L/K \), \( \rho : \text{Gal}(L/K) \rightarrow 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^* \) a fixed continuous character compatible with \( T \). Assume that \( A \) satisfies the following hypothesis:

1. \( Sh_p (K) \) is finite.
2. Every prime \( p \) of \( K \) for which \( A \) has bad reduction splits finitely in \( L \).
3. The reduction of \( A \) is semistable at every place of \( K \) dividing \( p \) and is an extension of an ordinary abelian variety by a torus for every \( \ell \in T \).
4. For every place \( t \in \Gamma \), the universal norm of \( A(L_t) \) is of finite index in \( A(K_t) \).

There is a p-adic height pairing \( <, >_p \) on \( A \) such that \( <, >_p := <, > \log_p \kappa(\gamma) \), where \( <, > \) is a p-adic height pairing defined by the author [p-adic heights for semistable abelian varieties\(^\text{\textsuperscript{\textlangle}}\), Compos. Math. (to appear)] and is equivalent to Schneider’s analytic height [P. Schneider, Invent. Math. 69, 401-409 (1982; Zbl 0509.14048)]. A necessary and sufficient condition for \( <, >_p \) to be nondegenerate is obtained. Further, define the groups \( \mathcal{I} := \text{Image}[H^1(\mathcal{O}_K, A^0_{\infty}) \rightarrow H^1(\mathcal{O}_K - T, A^0_{\infty})] \) and \( \mathcal{I}_\infty := \text{Image}[H^1(\mathcal{O}_L, A^0_{\infty}) \rightarrow H^1(\mathcal{O}_L - T, A^0_{\infty})] \). (They are quasi-isomorphic to the classical p-Selmer group of \( A \) over \( K \) and \( L \), respectively.) Write \( A_p = (A_p^0 - A_p^0(K)) \oplus A_p^{\text{inf}}(L) \) where \( A_p^{\text{inf}} \) is the divisible subgroup of \( A_p \). Then one can define \( A_p^{\text{fin}}(K) \) to be the \( K \)-rational points of \( A_p^{\text{fin}}(L) \). Define the \( \mathcal{L} \)-invariant of \( A \) with respect to \( L/K \) at a place \( \nu \in T \) by \( \mathcal{L}(A) := (A(K_p)/NA(K_p))/\text{Image}(\Phi(\mathcal{O}_K) \log_p \kappa(\gamma))^{\nu} \) and define the global \( \mathcal{L} \)-invariant of \( A \) with respect to \( L/K \) by \( \mathcal{L}(A) := \prod_{\nu \in T} \mathcal{L}(A) \).

The main result of the paper is to define a “good” \( \Lambda \)-module, \( H \), which is subject to a quasi-exact sequence

\[
0 \rightarrow \mathcal{I}_\infty \rightarrow H \rightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^\ell \rightarrow 0 \quad \text{or} \quad 0 \rightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^\ell \rightarrow H \rightarrow \mathcal{I}_\infty \rightarrow 0
\]

where \( \Gamma \) acts trivially on the \((\mathbb{Q}_p/\mathbb{Z}_p)^\ell \) term. Let \( f_H(t) = (t-1)^{\gamma} f_T(t) \), and define a p-adic L-function \( L_H(s) := f_H(\kappa(\gamma)^{1-s}) \). (This is a candidate for the p-adic L-function of an ordinary abelian variety \( A \) which is semistable at \( p \).) Let \( \rho = or_{\nu \in T} A_L(s) \) and \( r = \text{rank}_Z A(K) \). Then the main result of this paper is formulated in the following theorem:

One has \( \rho \geq r + e \). If \( <, >_p \) is nondegenerate, then \( \rho = r + e \) and the \( \rho \)-th derivative of \( L_H(s) \) has the
A functional equation for \( L(H) \) and \( b \) have the same \( p \)-norm.

where \( m_\ell \) denotes the number of connected components in the fibre of \( A \) over \( \ell \) and \( a \approx b \) means that \( a \) and \( b \) have the same \( p \)-norm.

A functional equation for \( L_H(s) \) is also proved. That is, \( f_H(t) = (-1)^\rho t^\rho f_H(1/t) \) where \( \lambda \) is the \( \lambda \)-invariant of \( H \) and \( \rho \) is the multiplicity of the root of 1 in \( f_H(t) \), and similarly, \( L_H(s) = (-1)^\rho \kappa(\gamma)^{N(1-s)} L_H(2-s) \). Several candidates for such a \( \lambda \)-module are tested, e.g., \( H^1(\Omega_L, A_{p=}) \), \( H^1(\Omega_L, A_{p=}) \), and Greenberg’s module.

**References:**

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**Keywords:**

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**MSC:**

14G10 Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture)
14K05 Algebraic theory of abelian varieties
14G40 Arithmetic varieties and schemes; Arakelov theory; heights
11G40 \( L \)-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture