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Iwasawa L-functions for multiplicative abelian varieties. (English) Zbl 0716.14008

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Iwasawa L-functions for abelian varieties with multiplicative reductions are studied, extending some results proved by *B. Mazur* in Invent. Math. 18, 183-266 (1972; Zbl 0245.14015) for abelian varieties with good ordinary reductions.

Let $p \neq 2$ be a prime, $\Gamma = \mathbb{Z}_p$ (as an additive topological group) with a generator γ, and $\Lambda := \lim_{\leftarrow \nu} \mathbb{Z}_p[\Gamma/p^\nu\Gamma]$. Then the map which sends T to γ-1 induces an isomorphism from $\mathbb{Z}_p[[T]]$ to Λ. The Iwasawa L-function for an elliptic curve was defined as the characteristic polynomial of the p-Selmer group of the curve. To generalize this definition to abelian varieties, one needs "good" Λ-modules which are finitely generated modules M over Λ. Such a module is quasi-isomorphic to the direct sum $\Lambda^\rho \oplus \mathbb{Z}/p^{\mu_i}\mathbb{Z}[[T]] \oplus (\oplus_j \mathbb{Z}_p[[T]](F_j)^{n_j})$ where ρ is the free rank of M, F_j is an irreducible distinguished polynomial for each j. The invariants $(\rho, \mu_i, \{F_j^{n_j}\})$ determine M completely up to quasi-isomorphism (i.e., up to finite kernel and cokernel). The μ-invariant of M is $\mu := \sum_i \mu_i$, the characteristic polynomial of M is $F_M(T) := p^\mu \prod_j (F_j(T))^{n_j}$ and $f_M(t)$ is the polynomial satisfying $f_M(T+1) = F_M(T)$.

Let K be a number field with ring of integers $\mathfrak{O}_K$. Let $A_{/K}$ be an abelian variety defined over K, A its Néron model over $\mathfrak{O}_K$, $\tilde{A}$ the dual abelian variety of A, A^0 the connected component of A and $A_{p^\infty} := \cup_\nu A_{p^\nu}$. Let Φ be defined by the short exact sequence $0 \to A^0 \to A \to \Phi \to 0$. Let L/K be a Γ-extension of K, T the set of all primes in K ramifying in L, $\log_p$ a p-adic logarithm of L/K, $\kappa : Gal(L/K) \to 1 + p^e\mathbb{Z}_p \subset \mathbb{Z}_p^*$ a fixed continuous character compatible with $\log_p$. If $\nu \in T$, e_ν denotes the dimension of a maximal split subtorus in the reduction of A at ν and $e := \sum_\nu e_\nu$.

Assume that A satisfies the following hypothesis:

1. $Sh_{p^\infty}(K)$ is finite.

2. Every prime of K for which A has bad reduction splits finitely in L.

3. The reduction of A is semistable at every place of K dividing p and is an extension of an ordinary abelian variety by a torus for every $t \in T$.

4. For every place $t \in T$, the universal norm of $A(L_t)$ is of finite index in $A(K_t)$.

There is a p-adic height pairing $<,>_p$ on A such that $<,>_p := <,>_\gamma \log_p \kappa(\gamma)$, where $<,>_\gamma$ is a p-adic height pairing defined by the author ["p-adic heights for semistable abelian varieties", Compos. Math. (to appear)] and is equivalent to Schneider's analytic height [*P. Schneider*, Invent. Math. 69, 401-409 (1982; Zbl 0509.14048)]. A necessary and sufficient condition for $<,>_p$ to be nondegenerate is obtained. Further, define the groups $\mathcal{I} := Image[H^1(\mathfrak{O}_K, A^0_{p^\infty}) \to H^1(\mathfrak{O}_K - T, A^0_{p^\infty})]$ and $\mathcal{I}_\infty := Image[H^1(\mathfrak{O}_L, A^0_{p^\infty}) \to H^1(\mathfrak{O}_L - T, A^0_{p^\infty})]$. (They are quasi-isomorphic to the classical p- Selmer group of A over K and L, respectively.) Write $A_{p^\infty}(L) = A^{\inf}_{p^\infty}(L) \oplus A^{fin}_{p^\infty}(L)$ where $A^{\inf}_{p^\infty}$ is the divisible subgroup of $A_{p^\infty}(L)$. Then one can define $A^{fin}_{p^\infty}(K)$ to be the K-rational points of $A^{fin}_{p^\infty}(L)$. Define the $\mathcal{L}_\nu$-invariant of A with respect to L/K at a place $\nu \in T$ by $\mathcal{L}_\nu(A) := (A(K_\nu)/NA(K_\nu))/(\Phi(\mathfrak{O}_{K_\nu})|\log_p \kappa(\gamma)|_p^{e_\nu})$ and define the global $\mathcal{L}$-invariant of A with respect to L/K by $\mathcal{L}(A) := \prod_{\nu \in T} \mathcal{L}_\nu(A)$.

The main result of the paper is to define a "good" Λ-module, H, which is subject to a quasi-exact sequence

$$0 \to \mathcal{I}_\infty \to H \to (\mathbb{Q}_p/\mathbb{Z}_p)^e \to 0 \quad or \quad 0 \to (\mathbb{Q}_p/\mathbb{Z}_p)^e \to H \to \mathcal{I}_\infty \to 0$$

where Γ acts trivially on the $(\mathbb{Q}_p/\mathbb{Z}_p)^e$ term. Let $f_H(t) = (t-1)^e f_{\mathcal{I}}(t)$, and define a p-adic L-function $L_H(s) := f_H(\kappa(\gamma)^{1-s})$. (This is a candidate for the p- adic L-function of an ordinary abelian variety A which is semistable at p.) Let $\rho = ord_{s=1} L_H(s)$ and $r = rank_{\mathbb{Z}} A(K)$. Then the main result of this paper is formulated in the following theorem:

One has $\rho \geq r + e$. If $< \ >_p$ is nondegenerate, then $\rho = r + e$ and the ρ-th derivative of $L_H(s)$ has the

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following value at $s = 1$:

$$L_H^{(\rho)}(1) \approx \mathcal{L}(A) \frac{\det < >_p Sh_K}{A_{p^\infty}^{fin}(K) \tilde{A}_{p^\infty}^{fin}(K)} \cdot \prod_{\ell \nmid \infty} m_\ell,$$

where m_ℓ denotes the number of connected components in the fibre of A over ℓ and $a \approx b$ means that a and b have the same p-norm.

A functional equation for $L_H(s)$ is also proved. That is, $f_H(t) = (-1)^\rho t^\lambda f_H(1/t)$ where λ is the λ-invariant of H and ρ is the multiplicity of the root of 1 in $f_H(t)$, and similarly, $L_H(s) = (-1)^\rho \kappa(\gamma)^{\lambda(1-s)} L_H(2-s)$. Several candidates for such a Λ-module are tested, e.g., $H^1(\mathfrak{O}_L, A^0_{p^\infty})$, $H^1(\mathfrak{O}_L, A_{p^\infty})$, and Greenberg's module.

Reviewer: N.Yui

MSC:

14G10 Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture)

14K05 Algebraic theory of abelian varieties

14G40 Arithmetic varieties and schemes; Arakelov theory; heights

11G40 L-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture

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functional equation for L-function; derivative of L-function; Birch and Swinnerton-Dyer conjecture; Iwasawa L-functions for abelian varieties with multiplicative reductions; p-adic height pairing

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