Barthel, Tobias; Heard, Drew; Valenzuela, Gabriel
Derived completion for comodules. (English) Zbl 1436.55018
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The aim of this paper is to study the generalization of local homology and derived completion to comodules over a Hopf algebroid (A, Ψ) with respect to an invariant ideal $I \lhd A$. The discrete case $\Psi = A$ corresponds to the usual setting of commutative algebra; working with comodules introduces new phenomena, for instance local homology can be non-zero in both negative and positive degrees.

The authors commence by I-adic completion of comodules in the non-derived setting. The subtlety is that the inverse limit of a diagram of Ψ-comodules is not in general created in A-modules. Under their hypothesis that (A, Ψ) is true-level, they give an explicit treatment of I-adic completion; this extends previous work of other authors.

They then turn to derived completion and local homology. For this, the derived category of Ψ-comodules is not adequate; the solution (under the appropriate hypotheses), based upon their earlier work [*T. Barthel* et al., Adv. Math. 335, 563–663 (2018; Zbl 1403.55008)], is to work with the monoidal stable ∞-category Stable_Ψ; this can be interpreted as passing from quasi-coherent to Ind-coherent sheaves. The I-torsion category $\mathrm{Stable}_\Psi^{I-\mathrm{tors}}$ is then defined as the localizing tensor ideal of Stable_Ψ generated by A/I.

They construct a local homology functor Λ^I for comodules and show that, in general, local homology cannot be calculated as the derived functors of completion. They give a criterion for a comodule to be Λ^I-local, which can be interpreted as a generalization of Bousfield and Kan's Ext p-completeness criterion.

Finally they consider I-torsion objects in the derived category of comodules as well as complete objects. There are at least three candidate stable ∞-categories of torsion modules; the main result relates these under the appropriate hypotheses. In the discrete case they characterize I-completion at the derived level, leading to a tilting-theoretic version of local duality.

This work is motivated by problems from stable homotopy theory, notably ongoing work on the algebraic chromatic splitting conjecture.

Reviewer: Geoffrey Powell (Angers)

MSC:

55P60 Localization and completion in homotopy theory
13D45 Local cohomology and commutative rings
14B15 Local cohomology and algebraic geometry
55U35 Abstract and axiomatic homotopy theory in algebraic topology

Keywords:

Hopf algebroid; comodule; local homology; derived completion

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