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**Symmetric matrices whose entries are linear functions.** (English. Russian original)

Zbl 1454.15023


The author considers matrices $A$ whose entries are linear functions in several real variables. In particular, he studies the problem of understanding when such matrices are positive (or negative) definite at some point. The author’s motivation mainly comes from semidefinite programming problems, where the search for a point at which a matrix is definite is important.

The author first defines the notion of a property that holds for almost every matrix and stresses that such a notion plays an important role in the development of heuristic algorithms. Given a set of matrices with entries depending on parameters $\xi_1, \xi_2, \ldots, \xi_m$, some property $\Phi(\xi_1, \xi_2, \ldots, \xi_m)$ holds for almost every matrix of this set if there exists a polynomial $p(\xi_1, \xi_2, \ldots, \xi_m)$, not identical to zero, such that for each set of values $\xi_1, \xi_2, \ldots, \xi_m$ if the property $\Phi(\xi_1, \xi_2, \ldots, \xi_m)$ does not hold, then the polynomial vanishes:

$$p(\xi_1, \xi_2, \ldots, \xi_m) = 0.$$ 

In the paper, the following results are proved:

1) For almost every $n \times n$ matrix $A$ whose entries are linear functions in $n$ variables, there exists a point at which the determinant of $A$ is positive and there exists another point at which the determinant of $A$ is negative. The same is true for almost every symmetric $n \times n$ matrix of the considered form.

2) For almost all sets of linear functions $\ell_0, \ell_1, \ldots, \ell_n$ in $n$ variables $x_1, x_2, \ldots, x_n$ and for any $n \times n$ matrix $M$, the matrix

$$\ell_0 M + \text{diag}(\ell_1, \ell_2, \ldots, \ell_n)$$

is definite at some point (here $\text{diag}(\ell_1, \ell_2, \ldots, \ell_n)$ denotes the diagonal matrix with entries $\ell_1, \ell_2, \ldots, \ell_n$ on the main diagonal). In particular, almost every symmetric $2 \times 2$ matrix whose entries are linear functions in two variables is definite at some point.

3) For almost all sets of linear functions $\ell_1, \ell_2, \ldots, \ell_n$ in $n$ variables $x_1, x_2, \ldots, x_n$ and for any numerical $n \times n$ matrix $M$, the matrix $M + \text{diag}(\ell_1, \ell_2, \ldots, \ell_n)$ is positive definite at some point and negative definite at another point.

4) For all triplets of linear functions $\ell_0, \ell_1, \ell_2$ in two variables $x_1$ and $x_2$, if $\ell_0(0, 0) \neq 0$, then the symmetric $3 \times 3$ matrix

$$\begin{bmatrix}
x_1 & \ell_0 & \ell_1 \\
\ell_0 & x_2 & \ell_2 \\
\ell_1 & \ell_2 & x_3
\end{bmatrix}$$

is definite at some point. The author also presents an example which shows that the condition $\ell_0(0, 0) \neq 0$ is essential.

At the end of the paper, the above results are illustrated using Hessian matrices of third-degree polynomials.

Reviewer: Janko Marovt (Maribor)

MSC:

15B57 Hermitian, skew-Hermitian, and related matrices
15A64 Matrices over function rings in one or more variables
05B20 Combinatorial aspects of matrices (incidence, Hadamard, etc.)
90C22 Semidefinite programming


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