Summary: The main result of this paper is the following: for all $b \in \mathbb{Z}$ there exists $k = k(b)$ such that

$$\max\{|A^{(k)}|,|(A + u)^{(k)}|\} \geq |A|^b,$$

for any finite $A \subset \mathbb{Q}$ and any nonzero $u \in \mathbb{Q}$. Here, $|A^{(k)}|$ denotes the $k$-fold product set $\{a_1 \cdots a_k : a_1, \ldots, a_k \in A\}$. Furthermore, our method of proof also gives the following $l_\infty$ sum-product estimate. For all $\gamma > 0$ there exists a constant $C = C(\gamma)$ such that for any $A \subset \mathbb{Q}$ with $|AA| \leq K|A|$ and any $c_1, c_2 \in \mathbb{Q} \setminus \{0\}$, there are at most $K^C|A|^\gamma$ solutions to

$$c_1 x + c_2 y = 1, \quad (x, y) \in A \times A.$$ 

In particular, this result gives a strong bound when $K = |A|^\epsilon$, provided that $\epsilon > 0$ is sufficiently small, and thus improves on previous bounds obtained via the Subspace Theorem.

In further applications we give a partial structure theorem for point sets which determine many incidences and prove that sum sets grow arbitrarily large by taking sufficiently many products.

We utilize a query-complexity analogue of the polynomial Freiman-Ruzsa conjecture, due to Pálvölgyi and Zhelezov (2020). This new tool replaces the role of the complicated setup of Bourgain and Chang (2004), which we had previously used. Furthermore, there is a better quantitative dependence between the parameters.

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