Summary: We generalize work of Deligne and Gillet-Soulé on a functorial Riemann-Roch type isometry, to the case of the trivial sheaf on cusp compactifications of Riemann surfaces of finite type  $\Gamma \setminus \mathbb{H}$, for $\Gamma \subset \text{PSL}_2(\mathbb{R})$ a Fuchsian group of the first kind, equipped with the Poincaré metric. This metric is singular at cusps and elliptic fixed points, and the original results of Deligne and Gillet-Soulé do not apply to this setting. Our theorem relates the determinant of cohomology of the trivial sheaf, with an explicit Quillen type metric defined in terms of the Selberg zeta function of $\Gamma$, to a metrized version of the $\psi$ tautological line bundle in the theory of moduli spaces of pointed orbicurves, and the self-intersection bundle of a suitable twist of the canonical sheaf $\omega$. We make use of surgery techniques through Mayer-Vietoris formulas for determinants of Laplacians, in order to reduce to explicit evaluations of such for model hyperbolic cusps and cones. We carry out these computations, which are of independent interest for theoretical physics, and can be adapted to other geometries. We go on to derive an arithmetic Riemann-Roch formula in the realm of Arakelov geometry, which applies in particular to integral models of modular curves with elliptic fixed points. This vastly extends previous work of the first author, whose deformation-theoretic methods were limited to torsion free Fuchsian groups. As an application, we treat in detail the case of the modular curve $X(1)$, and we show the interesting arithmetic content of the metrized $\psi$ line bundles. From this, we answer the longstanding question of evaluating the Selberg zeta special value $Z'(1, \text{PSL}_2(\mathbb{Z}))$. The result is expressed in terms of logarithmic derivatives of Dirichlet $L$ functions. In the analogy between Selberg zeta functions and Dedekind zeta functions of number fields, this formula can be seen as the analytic class number formula for $Z(s, \text{PSL}_2(\mathbb{Z}))$. The methods developed in this article were conceived so that they afford several variants, such as the determinant of cohomology of a flat unitary vector bundle with finite monodromies at cusps. Our work finds its place in the program initiated by Burgos-Kramer-Kühn of extending arithmetic intersection theory to singular Hermitian vector bundles.

MSC:

11M36 Selberg zeta functions and regularized determinants; applications to spectral theory, Dirichlet series, Eisenstein series, etc. (explicit formulas)
14G40 Arithmetic varieties and schemes; Arakelov theory; heights
11F72 Spectral theory; trace formulas (e.g., that of Selberg)
58J52 Determinants and determinant bundles, analytic torsion

Keywords:

Riemann-Roch; Arakelov geometry; Fuchsian group; Quillen metric; analytic surgery; Mayer-Vietoris; Dirichlet-to-Neumann; Selberg zeta function; L-functions

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References:


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