Fuhrmann, G.; Glasner, E.; Jäger, T.; Oertel, C.
Irregular model sets and tame dynamics. (English) [Zbl 07331134]

Assume that $X$ is a compact metric space. Let $(X, T, \Phi)$ be a topological dynamical system, that is, $T$ is a topological group and $\Phi$ a continuous left action of $T$ on $X$ by homeomorphisms on $X$. The notation $tx$ is used for the image $\Phi(t, x)$ of the action of $t \in T$ on $x \in X$. $(X, T)$ is called minimal if the orbit of every point $x \in X$ is dense in $X$, that is, $\text{Cl}(Tx) = X$.

We call a pair of closed and disjoint subsets $U_0, U_1 \subseteq X$ an independence pair if there exists an infinite set $S \subseteq T$ such that for all $a \in \{0, 1\}^S$ there is some $x \in X$ with $tx \in U_a(t \in S)$.

The following characterization of tameness is given in [D. Kerr and H. Li, Math. Ann. 338, No. 4, 869–926 (2007; Zbl 1131.46046), Proposition 6.4]: a topological dynamical system $(X, T)$ is non-tame if and only if there exists an independence pair.

Let $G$ be a non-compact, locally compact second countable abelian group. By the Birkhoff-Kakutani Theorem, $G$ is metrisable with a metric $d$ which can be chosen to be invariant under translations on the group.

A set $\Gamma \subseteq G$ is called $(r)$-uniformly discrete if there exists $r > 0$ such that $d(g, h) > r$ for all $g \neq h \in \Gamma$. $\Gamma$ is called $(R)$-relatively dense if there exists $R > 0$ such that $\Gamma \cap B_R(g) \neq \emptyset$ for all $g \in G$, where $B_R(g)$ denotes the $R$-ball centered at $g$. We call $\Gamma$ a Delone set if it is uniformly discrete and relatively dense.

Let $D(G)$ denote the space of Delone subsets of a given metrisable group $G$. $D(G)$ is a metric space with a metric from [J. Y. Lee et al., Ann. Henri Poincaré 3, No. 5, 1003–1018 (2002; Zbl 1025.37004)]. If $D_{r, R}(G) \subseteq D(G)$ denotes the subset of Delone sets that are $r$-uniformly discrete and $R$-relatively dense with fixed $r, R > 0$, then $(D_{r, R}(G), d)$ is a compact metric space. Under the condition FLC (finitary local complexity) on $\Gamma$, the dynamical hull, $\text{hull}(\Gamma) = \text{cl}(\Gamma - g | g \in G)$ is compact (see [M. Schlottmann, CRM Monogr. Ser. 13, 143–159 (2000; Zbl 0984.37005), Proposition 2.3]). The $G$-action $(\text{hull}(\Gamma), G)$ given by the translation is called a Delone dynamical system.

A cut and project scheme (CPS) consists of a triple $(G, H, L)$ of two locally compact abelian groups $G$ (called external group) and $H$ (internal group) and a co-compact discrete subgroup $L \subseteq G \times H$ such that the natural projections $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ satisfy: (i) the restriction $\pi_G|L$ is injective; (ii) the image $\pi_H(L)$ is dense.

As a consequence of (i), if we let $L = \pi_G(L)$ and $L^* = \pi_H(L)$, the star map $* : L \to L^* : l \to l^* = \pi_H \circ (\pi_G|L)^{-1}(l)$ is well defined and surjective. Given a compact set $W \subseteq H$ (referred to as window), it is defined the point set $\Lambda(W) = \pi_G(L \cap (G \times W)) = l \in L | l^* \in W$.

If $W$ is proper, i.e., $\text{Cl}(f W) = W$, then $\Lambda(W)$ is Delone set and has the FLC condition. In this case, we call $\Lambda(W)$ a model set, and $(\text{hull}(\Lambda(W)), G)$ becomes a Delone dynamical system. The window (and also the resulting model set) is called regular if $\mu_H(\partial W) = 0$, for the existing Haar measure on $H$, otherwise it is called irregular.

Theorem. Suppose that $\Lambda(W)$ is an irregular model set arising from a cut and project scheme $(G, H, L)$ with locally compact and second countable abelian groups $G$ and $H$ and cocompact discrete subgroup $L \subseteq G \times H$. Then there exists an infinite independence set for the dynamical hull, and consequently the translation action on the hull is not tame.

By examples, it is shown that even in the restrictive case of Euclidean cut and project schemes irregular model sets may be uniquely ergodic and have zero topological entropy. This provides negative answers to questions by M. Schlottmann [CRM Monogr. Ser. 13, 143–159 (2000; Zbl 0984.37005) and R. V. Moody (see [P. A. B. Pleasants and C. Huck, Discrete Comput. Geom. 50, No. 1, 39–68 (2013; Zbl 1307.37010)])].

Suppose $(X, T)$ and $(H, T)$ are topological dynamical systems. Then $(H, T)$ is called a factor of $(X, T)$ if there exists a continuous onto map $\beta : X \to H$ such that $\beta(tx) = t\beta(x)$ for all $t \in T$. A topological dynamical system $(H, T)$ is called a maximal equicontinuous factor (MEF) of $(X, T)$ if it is an equicontinuous factor of $(X, T)$ with the additional property that every other equicontinuous factor of $(X, T)$ is also a factor of $(H, T)$. The existence and the uniqueness of MEF is verified in [J. Auslander, Minimal
We call a group action \((X, T)\) almost automorphic if it is an almost one-to-one extension of its MEF, i.e.,
the set of injectivity points \(X_0 \subset X\) of \(\beta\) is dense in \(X\) and the MEF is minimal.

Given a topological dynamical system \((H, T)\), a Borel probability measure \(\mu\) on \(H\) is called \(T\)-invariant
if \(\mu(t\cdot x) = \mu(x)\) for all \(t \in T\). \((H, T)\) is called uniquely ergodic if there exists exactly one invariant measure \(\mu\).

If \((H, T)\) is equicontinuous and minimal, then \((H, T)\) is uniquely ergodic (see [E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory. Group representations. Springer, Cham (1963; Zbl 0115.10603), Theorem 15.13]).

An almost one-to-one extension \((X, T)\) of \((H, T)\) is called regular, if the projection \(H_0 = \beta(X_0)\) of the
set of injectivity points \(X_0\) of \(\beta\) has positive \(\mu\)-measure.

The following theorem solves the question raised by the second author in Problem 5.7 in [E. Glasner, Invent. Math. 211, No. 1, 213–244 (2018; Zbl 1384.54021)].

We finally claim the following result.

Theorem. Suppose that \((X, T)\) is an almost automorphic topological group action. If \((X, T)\) is tame, then
it is a regular extension of its maximal equicontinuous factor.

Reviewer: Nikita Shekutkovski (Skopje)

MSC:
37B05 Dynamical systems involving transformations and group actions with special properties (minimality, distality, proximality, expansivity, etc.)
22F05 General theory of group and pseudogroup actions
57M60 Group actions on manifolds and cell complexes in low dimensions

Keywords:
model sets; cut and project schemes; topological group actions; tame dynamics

Full Text: DOI arXiv

References:
34, 3, 259-272 (1980) · Zbl 0463.54039 · doi:10.1007/BF02760887


[17] Li, Jian; Tu, Singing; Ye, Xiangdong, Mean equicontinuity and mean sensitivity, Ergodic Theory Dynam. Systems, 35, 8, 2587-2612 (2015) · Zbl 1356.37016 · doi:10.1017/etds.2014.41


[28] außer H. Hauser, Relative topological entropy for actions of non-discrete groups on compact spaces in the context of cut and project schemes, preprint 2020, 1901.08985. · Zbl 1467.37020


[33] Paul, Michael E., Construction of almost automorphic symbolic minimal flows, General Topology and Appl., 6, 1, 45-56 (1976) · Zbl 0341.54054


[39] de Melo, Welington; van Strien, Sebastian, One-dimensional dynamics, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)] 25, xiv+605 pp. (1993), Springer-Verlag, Berlin · Zbl 0791.58003 · doi:10.1007/978-3-642-78043-1

[40] Kürka, Petr, Topological and symbolic dynamics, Cours Spécialisés [Specialized Courses] 11, xii+315 pp. (2003), Société Mathématique de France, Paris · Zbl 1038.37011


This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically
matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.