Ash, Avner; Yasaki, Dan
Steinberg homology, modular forms, and real quadratic fields. (English) Zbl 07334491
J. Number Theory 224, 323-367 (2021)

Let $R$ be a commutative ring, let $E$ be a real quadratic field, and let $\Gamma \subset \text{GL}_2(\mathbb{Z})$ be a congruence subgroup. The authors study the homomorphism $\psi_{\Gamma,E} := -\partial$, where $\partial : H_1(\Gamma, C) \to H_0(\Gamma, \text{St}(Q^2; R))$ is the connecting homomorphism arising from the short exact sequence

$$0 \to \text{St}(Q^2; R) \to \text{St}(E^2; R) \to C \to 0,$$

where $\text{St}$ denotes the Steinberg module. In the case $R = \mathbb{C}$, the authors prove that the image of $\psi_{\Gamma,E}$ can be described in terms of modular symbols. For general $R$, they prove that $\text{Im}(\psi_{\Gamma,E})$ always lies in a certain cuspidal subspace $H^c_0(\Gamma, \text{St}(Q^2; R))$. Using work of H. W. Lenstra jun. [Invent. Math. 42, 201–224 (1977; Zbl 0362.12012)], and assuming the Generalized Riemann Hypothesis (GRH), they prove that the cokernel of $\psi_{\Gamma,E}$ is a finitely-generated, torsion $R$-module (Theorem 9.3). For specific choices of $\Gamma$, the authors are able to prove stronger results (still assuming GRH). For example, in the case where $\Gamma = \Gamma_1(N)$ or $\Gamma_1(N)\pm$, it is shown that $\psi_{\Gamma,E}$ is surjective. The authors also indicate that unconditional versions of their results should follow from a suitably developed theory of so-called toral periods (at least in the case where $R = \mathbb{C}$), and they conclude by giving some numerical evidence.

Reviewer: Nils Matthes (Oxford)

MSC:
11F67 Special values of automorphic $L$-series, periods of automorphic forms, cohomology, modular symbols
11F75 Cohomology of arithmetic groups

Keywords:
arithmetic homology; Steinberg representation; real quadratic field; general linear group; arithmetic group; modular form

Software:
PFPK; ecdata; SageMath; Magma

Full Text: DOI arXiv

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