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Discrete subsets in topological groups and countable extremally disconnected groups. (English) [Zbl 07337077]

A. V. Arkhangel'skij [C. R. Acad. Sci., Paris, Sér. A 265, 822–825 (1967; Zbl 0168.43702)] asked whether ZFC is enough to prove the existence of a nondiscrete extremally disconnected topological group; and I. V. Protasov [Math. Notes 55, No. 1, 1 (1994; Zbl 0836.22003); translation from Mat. Zametki 55, No. 1, 150–151 (1994)] asked whether ZFC is enough to prove the existence of a countable nondiscrete topological group in which all discrete subsets are closed. (All topological groups are assumed to be Hausdorff.) In the present paper the authors give a negative answer to the second question, as well as a negative answer to the countable version of the first. A key ingredient is the notion of rapid filter, namely a free filter $F$ on $\omega$ such that every function from $\omega$ to itself is majorized by an increasing enumeration of a set in $F$.

Any filter extending a rapid filter is rapid as well, and it is known that ZFC cannot prove the existence of rapid ultrafilters. In answering the countable version of Arhangel'skii’s question, the authors show that the existence of a countable nondiscrete extremally disconnected topological group implies the existence of a rapid ultrafilter. And in answering Protasov’s question, the authors show that if $G$ is a countable nondiscrete topological group where the neighborhood filter of the identity $e$ is not rapid, then $G \setminus \{e\}$ contains a discrete subset whose only limit point is $e$. So in any model of ZFC in which no rapid ultrafilters exist, we have: (1) there is no countable nondiscrete extremally disconnected topological group; and (2) every countable nondiscrete topological group contains discrete subsets that are not closed.

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MSC:

54G05 Extremally disconnected spaces, F-spaces, etc.
54H11 Topological groups (topological aspects)
03E35 Consistency and independence results
22A05 Structure of general topological groups

Keywords:
discrete subsets; extremely disconnected; countable topological groups; rapid filters

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References:
