Xu, Yi-Jing; **Yau, Stephen S.-T.**

A sharp estimate of the number of integral points in a tetrahedron. (English) Zbl 0734.11048

J. Reine Angew. Math. 423, 199-219 (1991).

The general problem of counting the number Q of nonnegative integral points satisfying

$$(1) \quad x/a + y/b + z/c \leq 1$$

where a,b, and c are positive integers, has been a challenge for many years. In this paper, the number P of positive integral points satisfying (1) where a,b and c are positive real numbers, is counted. Of course, one can deduce the estimate of Q once the estimate of P is known. The novelty in the problem is that the lattice points in a polytope whose vertices are not necessarily integer points (even rational points) are counted. Previous appearance of this sort of count, starting with Ramanujan and Hardy, and running through Spencer and Beukers, are entirely asymptotic. The purpose of this paper is to provide the best upper estimate for the number P. Theorem: Let $a \geq b \geq c \geq 2$ be real numbers. Let P be the number of positive integral solutions of (1) i.e. $P = \#\{(x,y,z) \in \mathbb{Z}_3^+ : x/a + y/b + z/c \leq 1\}$ where $\mathbb{Z}_+$ is the set of positive integers. Then

$$6P \leq (a-1)(b-1)(c-1) - c + 1$$

and the equality is attained if and only if $a = b = c = integers$.

Corollary: Let $a \geq b \geq c \geq 1$ be real numbers. Let Q be the number of nonnegative integral points satisfying $x/a + y/b + z/c \leq 1$. Then

$$Q \leq (s^2 + (a+b)s)/6abc$$

where $s = abc + ab + ac + bc$ and the equality is attained if and only if $a = b = c = integers$.

Reviewer: Yi-Jing Xu

MSC:

11P21 Lattice points in specified regions

Keywords:

lattice points in a polytope with real vertices; best upper estimate

Full Text: Crelle EuDML